Lipschitz stability of canonical Jordan bases of H-selfadjoint matrices

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References

[1] V. Olshevsky, *Change of Jordan structure of G-selfadjoint operators and selfadjoint operator functions under small perturbations*, (in Russian), Izvestia Akad. nauk U.S.S.R., 54 (No. 5) (1990), 1021 - 1048. English translation : AMS, Math. U.S.S.R. Izvestia, 37 (No. 2) (1991), 371 - 396.

[2] **T.Bella, V. Olshevsky, U.Prasad**, ***Lipschitz stability of canonical Jordan bases of H-selfadjoint matrices under structure-preserving perturbations*, Accepted to Linear Algebra and Its Applications, available online Feb. 2008.**

[3] (with L.Rodman) *Lipschitz stability of* **real** *canonical Jordan bases of H-selfadjoint matrices under structure-preserving perturbations*, in preparation.

Small perturbation problems

- Consider

Fixed $A_0 \in \mathbb{C}^{n \times n} \quad \longrightarrow \quad A \in \mathbb{C}^{n \times n}$, so that $\|A - A_0\|$ is sufficiently **small**.

- What can happen with
  - eigenvalues
  - Jordan structure
  - **Jordan bases**

Vast literature

Monographs by Ostrowski, Kato, Stewart et al, Gohberg et al, Mehrman et al, Bhatia, Markus, and **many others**.

What happens with eigenvalues? Example 1. Splitting.

$$A_0 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \rightarrow \quad A = \begin{bmatrix} \mu_1 & 1 \\ 0 & \mu_2 \end{bmatrix}$$

$$\sigma(A_0) = \{\lambda\} \quad \rightarrow \quad \sigma(A) = \{\mu_1, \mu_2\}$$

Lipschitz-type bound

$$|\lambda - \mu_i| \leqslant K \left\| \begin{bmatrix} \lambda - \mu_1 & 0 \\ 0 & \lambda - \mu_2 \end{bmatrix} \right\| = K\|A - A_0\|$$

Do we always have such Lipschitz-type bounds? No.

Example2. Companion matrices.

$$\det(xI - \begin{bmatrix} 0 & 0 & -\mathbf{a_0} \\ 1 & 0 & -\mathbf{a_1} \\ 0 & 1 & -\mathbf{a_2} \end{bmatrix}) = \mathbf{a_0} + \mathbf{a_1}x + \mathbf{a_2}x^2 + x^3.$$

$$\det(xI - \begin{bmatrix} 0 & 0 & \boxed{0} \\ 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \end{bmatrix}) = x^3 \quad \longrightarrow \quad \det(xI - \begin{bmatrix} 0 & 0 & \boxed{\epsilon} \\ 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \end{bmatrix}) = x^3 - \boxed{\epsilon}$$

$$\sigma(A_0) = \{\lambda = \mathbf{0}\} \quad \longrightarrow \quad \sigma(A) = \{\mu_{\mathbf{1}} = \epsilon^{\mathbf{1/3}}, \mu_{\mathbf{2}} = \epsilon^{\mathbf{1/3}}, \mu_{\mathbf{3}} = \epsilon^{\mathbf{1/3}}\}$$

The bound is NOT of the Lipschitz type.

$$|\lambda - \mu_{\mathbf{k}}| = |\mathbf{0} - \epsilon^{\mathbf{1/3}}| = K\|A - A_0\|^{\mathbf{1/3}}$$

What if the single eigenvalue does not split? Lipschitz-type bound

If $\sigma(A_0) = \{\lambda\}$ and $\sigma(A) = \{\mu\}$, then

$$|\lambda - \mu| = \frac{1}{n}|tr(A_0) - tr(A)| \leqslant K\|A - A_0\|.$$

- What can happen with
  - eigenvalues ✓
  - Jordan structure
  - **Jordan bases**

What can happen with the Jordan structure if the eigenvalues do not split?

Single eigenvalue. No splitting. Full description. Gohberg-Kaashoek(1978), Den Boer-Thijsse (1980), Markus-Parilis (1980), O.-Matsaev(1994)

$\sigma(A_0) = \{\lambda\}$	$\leftrightarrow$	$\underbrace{m_1(A_0,\lambda) \geqslant m_2(A_0,\lambda) \geqslant \dots \geqslant 0 \geqslant \dots \geqslant 0}_{n \text{ Jordan blocks}}$

If $\sigma(A_0) = \{\lambda\}$ and $\sigma(A) = \{\mu\}$ and $\|A - A_0\|$ is sufficiently **small**, then

$$\boxed{\{m_k(A_0,\lambda)\} \prec \{m_k(A_0,\lambda)\} \qquad \textbf{(majorization)}}$$

i.e.,

$$m_1(A_0,\lambda) \leqslant m_1(A,\mu)$$

$$m_1(A_0,\lambda) + m_2(A_0,\lambda) \leqslant m_1(A,\mu) + m_2(A,\mu)$$

$$m_1(A_0,\lambda) + m_2(A_0,\lambda) + m_3(A_0,\lambda) \leqslant m_1(A,\mu) + m_2(A,\mu) + m_3(A,\mu)$$

What can happen with the Jordan bases if the eigenvalues do not split and the Jordan structure does not change much?

- What can happen with
  - eigenvalues ✓
  - Jordan structure ✓
  - **Jordan bases**

Theorem. (Lipschitz stability of Jordan chains.) Let $\lambda \in \sigma(A_0)$, and let Γ be a small circle containing inside only one eigenvalue λ of A_0. Let $\{f_r^{(k)}\}_{r=0}^{m_k(A_0,\lambda)-1}\}$ be a **fixed** Jordan basis in the root subspace of A_0 corresponding to λ.

There exists a small $\epsilon > 0$ such that $\forall A$ with $\|A - A_0\| \leqslant \epsilon$ the following statement holds. If A has only one eigenvalue inside Γ, say, μ, and

$$m_k(A_0, \lambda) = m_k(A, \mu), \qquad k = 1, 2, \ldots, d \qquad \textbf{(for some } d\textbf{)}$$

then there exists a Jordan basis $\{\{g_r^{(k)}\}_{r=0}^{m_k(A,\mu)-1}\}$ (in the root subspace of A corresponding to μ) such that

$$\|g_r^{(k)} - f_r^{(k)}\| \leqslant K\|A - A_0\| \qquad k = 1, 2, \ldots, \mathbf{d}.$$

The constant $K > 0$ depends on **fixed** A_0 and **fixed** $\{f_r^{(k)}\}_{r=0}^{m_k(A_0,\lambda)-1}\}$ only.

Example

sizes of Jordan blocks of A_0	Jordan chains of A_0	sizes of Jordan blocks of A	Jordan chains of A
$m_1(A_0,\lambda) = 4$	$f_0^{(1)}, f_1^{(1)}, f_2^{(1)}, f_3^{(1)}$	$m_1(A,\mu) = 4$	$g_0^{(1)}, g_1^{(1)}, g_2^{(1)}, g_3^{(1)}$
$m_2(A_0,\lambda) = 3$	$f_0^{(2)}, f_1^{(2)}, f_2^{(2)}$	$m_2(A,\mu) = 3$	$g_0^{(2)}, g_1^{(2)}, g_2^{(2)}$
$m_3(A_0,\lambda) = 2$	$f_0^{(3)}, f_1^{(3)}$	$m_3(A,\mu) = 3$	$g_0^{(3)}, g_1^{(3)}, g_2^{(3)}$
$m_4(A_0,\lambda) = 2$	$f_0^{(4)}, f_1^{(4)}$	$m_4(A,\mu) = 1$	$g_0^{(4)}$

In this example only the first two Jordan chains are Lipschitz stable:

$$\|f_k^{(1)} - g_k^{(1)}\| \leqslant K\|A - A_0\|, \qquad \|f_k^{(2)} - g_k^{(2)}\| \leqslant K\|A - A_0\|$$

Corollary. Lipschitz stability of similarity matrices.

Let A_0 be fixed, and let A be a matrix similar to A_0. Then a (highly nonunique) similarity matrix A can be chosen to satisfy

$$\|I - S\| \leqslant K\|A - A_0\|.$$

The constant $K > 0$ depends on A_0 only.

H-self–adjoint matrices

- Let H be Hermitian and invertible (not necessarily positive definite). The **indefinite inner product** is defined via

$$[x, y]_H = y^* H x$$

-

A is self–adjoint with respect to a Euclidean Inner Product	A is **self–adjoint** with respect to an **Indefinite** Inner Product
$(Ax, y) = (x, Ay)$	$[Ax, y]_H = [x, Ay]_H$
$A = A^*$	$HA = A^*H$
Hermitian	H**-self–adjoint**

- Applications: see **[GLR05]**.

The relation "$(\cdot,\cdot) \xrightarrow{S} (\cdot,\cdot)$"

- We will use the notations

$$(A,H) \xrightarrow{S} (B,G) \text{ to mean that } \boxed{S^{-1}AS} = B \text{ and } \boxed{S^{*}HS} = G.$$

- A simple calculation shows that if A is **H-selfadjoint** and $(A,H) \xrightarrow{S} (B,G)$, then B is **G-selfadjoint**. Indeed,

$$HA = A^{*}H \Longrightarrow \mathbf{S}^{*}HA\mathbf{S} = \mathbf{S}^{*}A^{*}H\mathbf{S} \Longrightarrow \underbrace{\boxed{\mathbf{S}^{*}H\mathbf{S}}}_{G}\underbrace{\boxed{\mathbf{S}^{-1}A\mathbf{S}}}_{B} = \underbrace{\boxed{\mathbf{S}^{*}A^{*}\mathbf{S}^{-*}}}_{B^{*}}\underbrace{\boxed{\mathbf{S}^{*}H\mathbf{S}}}_{G}$$

Weirstrass theorem. Canonical Jordan basis

Definition. **SIP** matrix:

$$P = \begin{bmatrix} 0 & \cdots & \cdots & 0 & \mathbf{1} \\ \vdots & & \iddots & \mathbf{1} & 0 \\ \vdots & 0 & \iddots & 0 & \vdots \\ 0 & \mathbf{1} & \iddots & & \vdots \\ \mathbf{1} & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

Weirstrass theorem. Let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint. Then there exists an invertible matrix S such that $(A, H) \xrightarrow{S} (J, P)$ where

$$J = \underbrace{J(\lambda_1) \oplus \cdots \oplus J(\lambda_\alpha)}_{\text{real Jordan blocks}} \oplus \underbrace{\widetilde{J}(\lambda_{\alpha+1}) \oplus \cdots \oplus \widetilde{J}(\lambda_\beta)}_{\text{non-real Jordan blocks}}$$

is a **Jordan matrix**, and

$$P = \underbrace{\epsilon_1 P_1 \oplus \cdots \oplus \epsilon_\alpha P_\alpha}_{\text{signed sip matrices}} \oplus \underbrace{P_{\alpha+1} \oplus \cdots \oplus P_\beta}_{\text{unsigned sip matrices}}$$

Definition. The set of signs $\{\epsilon_k = \pm 1\}$ is called **sign characteristic**.

Example of a Canonical Jordan basis. Flipped orthogonality

$$J = \left[\begin{array}{ccc|cc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right], \qquad P = \left[\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

Definition. Canonical basis. The Jordan basis $\{\{e_1, e_2, e_3\}, \{e_4, e_5\}\}$ of (J, P)

$$0 \leftarrow e_1 \leftarrow e_2 \leftarrow e_3, \qquad 0 \leftarrow e_4 \leftarrow e_5.$$

is called **canonical**.

In this example the **sign characteristic** is: $\epsilon_1(J, 2) = +1, \quad \epsilon_1(J, 3) = -1$.

Structure–preserving perturbations of matrices self–adjoint with respect to an indefinite inner products

➠ **Theorem.** Let $A_0 \in \mathbb{C}^{n\times n}$ be a fixed H_0-selfadjoint matrix. Let $\lambda \in \sigma(A_0)$, and let Γ be a small circle containing inside only one eigenvalue λ of A_0. Let

$$\{\{f_r^{(k)}\}_{r=0}^{m_k(A_0,\lambda)-1}\}$$

be a **fixed canonical** Jordan basis in the root subspace of A_0 corresponding to λ. There exists a small $\epsilon > 0$ such that for any H-selfadjoint matrix A with

$$\|A - A_0\| + \|H - H_0\| \leqslant \epsilon$$

the following statement holds. If A has only one eigenvalue inside Γ, say, μ, and

$$m_k(A_0,\lambda) = m_k(A,\mu), \qquad k = 1,2,\ldots,d \qquad \textbf{(for some } d\textbf{)}$$

then there exists a **canonical** Jordan basis $\{\{g_r^{(k)}\}_{r=0}^{m_k(A,\mu)-1}\}$ (in the root subspace of A corresponding to μ) such that

$$\|g_r^{(k)} - f_r^{(k)}\| \leqslant K(\|A - A_0\| + \|H - H_0\|) \qquad k = 1,2,\ldots,\mathbf{d}.$$

The constant $K > 0$ depends on **fixed** A_0 and **fixed** $\{\{f_r^{(k)}\}_{r=0}^{m_k(A_0,\lambda)-1}\}$ only.

Structure–preserving perturbations of matrices self–adjoint with respect to an indefinite inner products

➠ **Corollary.** Let $A_0 \in \mathbb{C}^{n\times n}$ be a fixed H_0-selfadjoint matrix. There exist constants $K, \delta > 0$ (depending on A_0 and H_0 only) such that the following assertion holds. For any H-selfadjoint matrix A similar to A_0 and satisfying

$$\|A - A_0\| + \|H - H_0\| < \delta,$$

there exists an invertible matrix S such that

$$S^{-1}AS = A_0 \qquad \text{and} \qquad S^{*}HS = H_0$$

satisfying

$$\|I - S\| \leqslant K\left(\|A - A_0\| + \|H - H_0\|\right).$$

Partial stablity of sign characteristic

- **Theorem (Gohberg, Lancaster, Rodman).** Let $A_0 \in \mathbb{C}^{n\times n}$ be H_0-selfadjoint. Let Γ be a small circle containing inside only one eigenvalue λ of A_0.

  There exists a small $\epsilon > 0$ such that for any H-selfadjoint matrix A with

  $$\|A - A_0\| + \|H - H_0\| \leqslant \epsilon$$

  the following statement holds. If A has only one eigenvalue inside Γ, say, μ, and

  $$m_k(A_0, \lambda) = m_k(A, \mu), \qquad k = 1, 2, \ldots, n$$

  then A_0 and A have the same sign characteristic:

  $$\epsilon_k(A_0, \lambda) = \epsilon_k(A, \mu), \qquad k = 1, 2, \ldots, n$$

- **Refinement.** If

  $$m_k(A_0, \lambda) = m_k(A, \mu), \qquad k = 1, 2, \ldots, d \qquad \textbf{(for some } d < n\textbf{)}$$

  then A_0 and A have the same sign characteristic:

  $$\epsilon_k(A_0, \lambda) = \epsilon_k(A, \mu), \qquad k = 1, 2, \ldots, d$$

Example

sizes of Jordan blocks of A_0	**canonical** Jordan chains of H_0-s/a matrix A_0	sizes of Jordan blocks of A	**canonical** Jordan chains of H-s/a matrix A
$m_1(A_0, \lambda) = 4$	$f_0^{(1)}, f_1^{(1)}, f_2^{(1)}, f_3^{(1)}$	$m_1(A, \mu) = 4$	$g_0^{(1)}, g_1^{(1)}, g_2^{(1)}, g_3^{(1)}$
$m_2(A_0, \lambda) = 3$	$f_0^{(2)}, f_1^{(2)}, f_2^{(2)}$	$m_2(A, \mu) = 3$	$g_0^{(2)}, g_1^{(2)}, g_2^{(2)}$
$m_3(A_0, \lambda) = 2$	$f_0^{(3)}, f_1^{(3)}$	$m_3(A, \mu) = 3$	$g_0^{(3)}, g_1^{(3)}, g_2^{(3)}$
$m_4(A_0, \lambda) = 2$	$f_0^{(4)}, f_1^{(4)}$	$m_4(A, \mu) = 1$	$g_0^{(4)}$

In this example only the first two **canonical** Jordan chains are Lipschitz stable:

$$\|f_k^{(1)} - g_k^{(1)}\| \leqslant K\|A - A_0\| + \|H - H_0\|, \qquad \|f_k^{(2)} - g_k^{(2)}\| \leqslant K\|A - A_0\| + \|H - H_0\|$$

In this example only the first two **signs** of the sign characteristic are stable:

$$\epsilon_1(A_0, \lambda) = \epsilon_1(A, \mu), \qquad \epsilon_2(A_0, \lambda) = \epsilon_2(A, \mu)$$

Part 2. Hadamard-Sylvester vs Pseudo-Noise matrices

[4] T.Bella, V.Olshevsky, L. Sakhnovich, Ranks of Hadamard Matrices and Equivalence of Sylvester Hadamard and Pseudo-Noise Matrice, In Recent Advances in Matrix and Operator Theory, Operator Theory: Advances and Applications, Volume 179, Birkhauser Basel, 2008.

Hadamard Matrices

Hadamard matrices of size n x n, are $(-1, 1)$ matrices such that

$$H_n^T H_n = nI_n$$

A special case: **Hadamard-Sylvester matrices**

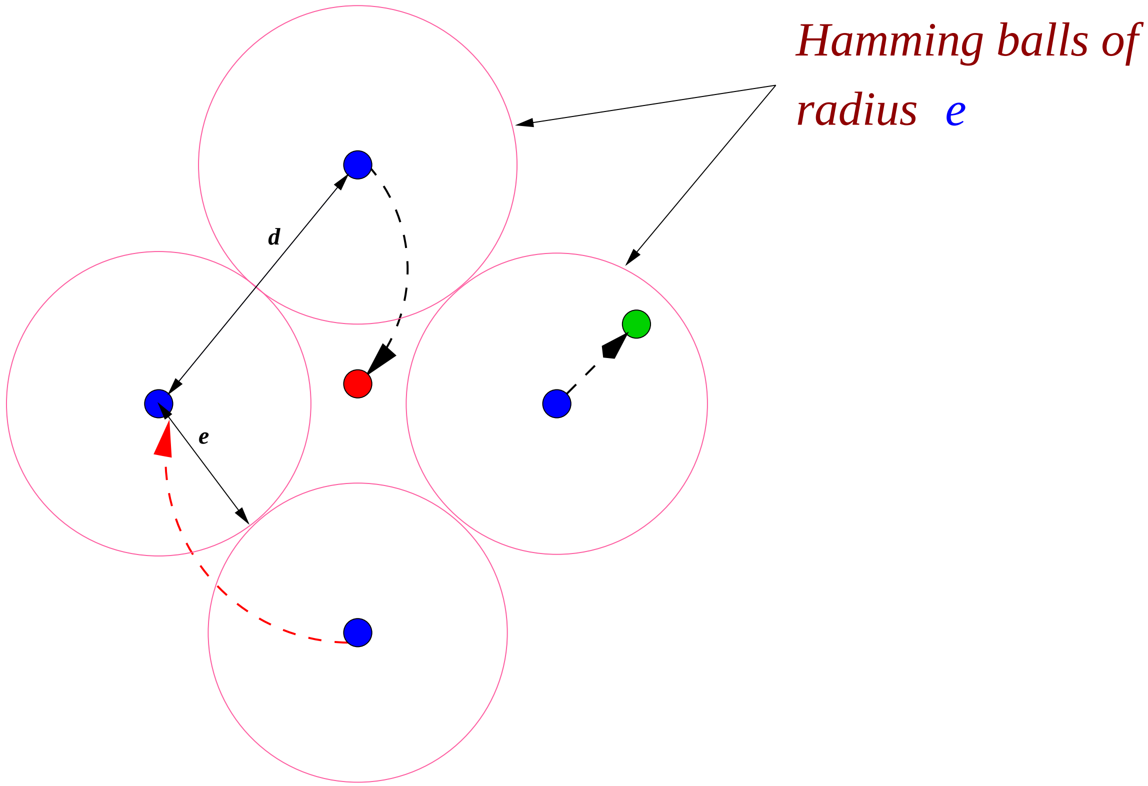
$$H_1 = [1], \qquad H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$$

For example,

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

What makes Hadamard-Sylvester Matrices to be Useful for Coding?

- **Rows & Columns Orthogonal** - Any two rows/columns of an $n \times n$ matrix agree in exactly $\frac{n}{2}$ places.
- The **minimum distance** between the columns is large: $\frac{n}{2}$



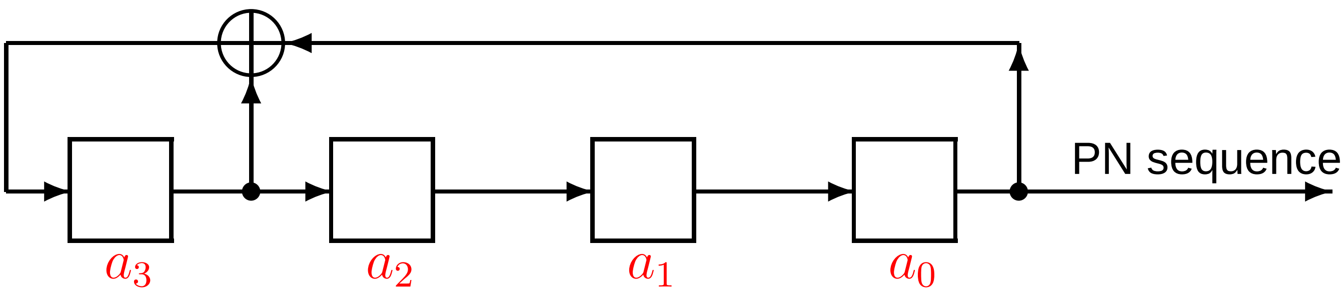
- This code is capable of **correcting** up to $\frac{n-2}{4}$ errors.

Another good code: the columns of Pseudo-Noise Matrices

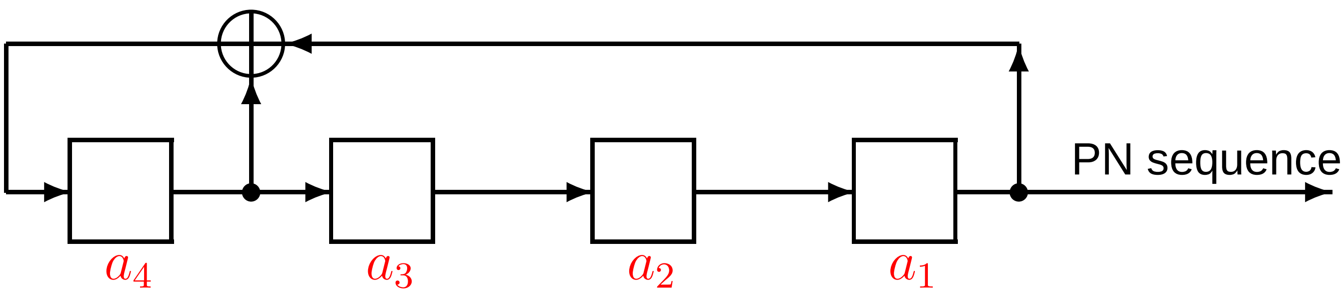
- **Primitive feedback registers. Example for $n = 4$**

$$\mathbf{a_i} = \mathbf{a_{i-1}h_3} + \mathbf{a_{i-2}h_2} + \mathbf{a_{i-3}h_1} + \mathbf{a_{i-4}h_0}$$

- **Time moment zero**. The **initial** state $\{a_3, a_2, a_1, a_0\}$:



- **Time moment one**. The **next** state $\{a_4, a_3, a_2, a_1\}$:



- A register of length m can have **at most** $2^m - 1$ **different states** (**could be less**).
- A register is called **primitive** if it passes through **all possible** $2^m - 1$ **states**.

PN Sequences

- The output

$$\mathbf{a_0 a_1 a_2} \ldots$$

  of a primitive register is called a PN sequence.

- **Fact**: $\forall\, m\, \exists$ primitive registers.

- **Fact**: A PN sequence generated by an **m**-degree primitive register is **periodic** with period $2^m - 1$.

- For $h(x) = x^4 + x^3 + 1$ (i.e., $m = 4$), and the initial state $a_0 a_1 a_2 a_3 = 1000$, the resulting PN Sequence is given by

$$\underbrace{1000111101011 00}_{\text{period 15}}\underbrace{100011110101100}_{\text{period 15}}\underbrace{100011110101100}_{\text{period 15}}\ldots\ldots$$

PN Matrices

- A **Pseudo Noise Matrix** is one of the form

$$T = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \widetilde{T} & \\ 0 & & & \end{bmatrix}$$

where $\widetilde{T}$ is a **circulant Hankel** matrix whose rows are **PN sequences**.

- **Theorem**

The $(0, 1)$ Hadamard-Sylvester matrices and the $(0, 1)$ PN matrices are equivalent, i.e., they can be obtained one from another via row and column permutations.

- **Sakhnovich(1998)** proved this result for $n = 16$ using combinatorial tricks.