Orthogonal Polynomials, Moments, Measure Deformation, Dynamical Systems, and SVD Algorithm

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February 29, 2008 @ Golub Memorial Conference

Outline

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Basic Dynamics

- Lax dynamics:

$$\begin{aligned} \frac{dX(t)}{dt} &:= [X(t), k_1(X(t))] \\ X(0) &:= X_0. \end{aligned}$$

- Parameter dynamics:

$$\begin{aligned} \frac{dg_1(t)}{dt} &:= g_1(t)k_1(X(t)) \\ g_1(0) &:= I. \end{aligned}$$

and

$$\begin{aligned} \frac{dg_2(t)}{dt} &:= k_2(X(t))g_2(t) \\ g_2(0) &:= I. \end{aligned}$$

- $k_1(X) + k_2(X) = X$.

Similarity Property

$$X(t) = g_1(t)^{-1} X_0 g_1(t) = g_2(t) X_0 g_2(t)^{-1}.$$

- Define $Z(t) = g_1(t)X(t)g_1(t)^{-1}$.
- Check

$$\begin{aligned}\frac{dZ}{dt} &= \frac{dg_1}{dt} X g_1^{-1} + g_1 \frac{dX}{dt} g_1^{-1} + g_1 X \frac{dg_1^{-1}}{dt} \\ &= (g_1 k_1(X)) X g_1^{-1} + g_1 (X k_1(X) - k_1(X) X) g_1^{-1} \\ &\quad + g_1 X(-k_1(X) g_1^{-1}) = 0.\end{aligned}$$

- Thus $Z(t) = Z(0) = X(0) = X_0$.

Decomposition Property

$$exp(tX_0) = g_1(t)g_2(t).$$

- Trivially $exp(X_0 t)$ satisfies the IVP

$$\frac{dY}{dt} = X_0 Y,\ Y(0) = I.$$

- Define $Z(t) = g_1(t)g_2(t)$.
- Then $Z(0) = I$ and

$$\begin{aligned} \frac{dZ}{dt} &= \frac{dg_1}{dt} g_2 + g_1 \frac{dg_2}{dt} \\ &= (g_1 k_1(X))g_2 + g_1(k_2(X)g_2) = g_1 X g_2 \\ &= X_0 Z \ \text{ (by Similarity Property).} \end{aligned}$$

- By the uniqueness theorem in ODEs, $Z(t) = exp(X_0 t)$.

Reversal Property

$$exp(tX(t)) = g_2(t)g_1(t).$$

- By Decomposition Property,

$$\begin{aligned} g_2(t)g_1(t) &= g_1(t)^{-1}exp(X_0t)g_1(t) \\ &= exp(g_1(t)^{-1}X_0g_1(t)t) \\ &= exp(X(t)t). \end{aligned}$$

Abstract *QR*-type Decomposition

- Arbitrary subspace decomposition $gl(n) \Longleftrightarrow$ Factorization of a *one-parameter semigroup* in the neighborhood of I as the product of two nonsingular matrices , i.e.,

$$exp(X_0 t) = g_1(t) g_2(t).$$

  - Lie algebra decomposition of $gl(n) \Longleftrightarrow$ Lie group decomposition of $Gl(n)$ in the neighborhood of I.
- The product $g_1(t)g_2(t)$ will be called the *abstract $g_1 g_2$ decomposition* of $exp(X_0 t)$.

Abstract QR-type Algorithm

- By setting $t = 1$, we have

$$\begin{aligned} exp(X(0)) &= g_1(1)g_2(1) \\ exp(X(1)) &= g_2(1)g_1(1). \end{aligned}$$

- The dynamical system for $X(t)$ is autonomous $\Longrightarrow$ The above phenomenon will occur at every feasible integer time.
- Corresponding to the abstract $g_1 g_2$ decomposition, the above iterative process for all feasible integers will be called the *abstract $g_1 g_2$ algorithm*.

Toda Flow

- Lie algebra decomposition:
  - $gl(n) = \{\text{skew symmetric}\} \bigoplus \{\text{upper triangular}\}$.
  - Write
    $$X = X^o + X^+ + X^-$$
    where X^o is the diagonal, X^+ the strictly upper triangular, and X^- the strictly lower triangular part of X.
  - Define
    $$\Pi_0(X) := X^- - X^{-\top}.$$
- The Toda lattice (Symes'82, Deift el al'83):

$$\begin{aligned} \frac{dX}{dt} &= [X, \Pi_0(X)] \\ X(0) &= X_0. \end{aligned}$$

QR Algorithm and Limiting Behavior

- Sampled at integer times, $\{X(k)\}$ gives the same sequence as does the *QR* algorithm applied to the matrix $A_0 = exp(X_0)$.
- Evolution starting from X_0,
  - The flow maintains the spectrum.
  - The construction of the Toda lattice is based on the physics.
    - This is a Hamiltonian system.
    - A certain physical quantities are kept at constant, i.e., this is a *completely integrable* system.
    - Asymptotic behavior can be analyzed via ODE theory.

Orthogonal Polynomials

- Orthogonal polynomials with respect to a given a measure $\mu(x)$,

$$\int p_k(x)p_\ell(x)\,d\mu(x) = \delta_{k,\ell}, \quad k,\ell = 0,1,\ldots.$$

- Three-term recurrence relationship.

$$xp_k(x) = a_k p_{k+1}(x) + b_k p_k(x) + a_{k-1}p_{k-1}(x), \quad k = 1,2,\ldots,$$

  with $p_{-1}(x) \equiv 0$ and $p_0(x) \equiv 1$.

- Corresponding monic polynomials $\{\tilde{p}_k(x)\}_k$,

$$x\tilde{p}_k(x) = \tilde{p}_{k+1} + b_k\tilde{p}_k(x) + a_{k-1}^2\tilde{p}_{k-1}(x).$$

Hankel Determinants

$$H_k := \det \begin{bmatrix} s_0 & s_1 & \dots & s_{k-1} \\ s_1 & s_2 & & s_k \\ \vdots & & & \vdots \\ s_{k-1} & s_k & \dots & s_{2k-2} \end{bmatrix}.$$

- s_j are moments with respect to μ,

$$s_j := \int x^j \, d\mu(x), \quad j = 0, 1, \dots.$$

Classical Moment Problem

- $\tilde{p}_k(x)$ is given by (Akhiezer'65,Szegö'75),

$$\begin{aligned}\tilde{p}_k(x) &= \frac{1}{H_k}\det\begin{bmatrix} s_0 & s_1 & \dots & s_k \\ s_1 & s_2 & & s_{k+1} \\ \vdots & & & \vdots \\ s_{k-1} & s_k & \dots & s_{2k-1} \\ 1 & x & \dots & x^k \end{bmatrix}, \\ &= x^k + c_1^{(k)}x^{k-1} + \dots + c_{k-1}^{(k)}x + c_k^{(k)},\end{aligned}$$

-

$$c_j^{(k)} = \frac{(-1)^j}{H_k}\det\begin{bmatrix} s_0 & \dots & s_{k-j-1} & s_{k-j+1} & \dots & s_k \\ s_1 & & & & & s_{k+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ s_{k-1} & \dots & s_{2k-j-2} & s_{2k-j} & \dots & s_{2k-1} \end{bmatrix}.$$

Polynomials in Moments

- By comparing the corresponding coefficients,

$$
\begin{aligned}
a_k^2 &= \frac{H_k H_{k+2}}{H_{k+1}^2}, \\
b_k &= c_1^{(k)} - c_1^{(k+1)}.
\end{aligned}
$$

- A classical result.

$$\mu(x) \leftrightsquigarrow s_j \leftrightsquigarrow H_k \leftrightsquigarrow p_k(x).$$

Measure Deformation

- Rewrite orthogonality in matrix form,

$$\underbrace{\begin{bmatrix} b_0 & a_0 & 0 & & & \\ a_0 & b_1 & a_1 & 0 & & \\ 0 & a_1 & b_2 & a_2 & 0 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}}_{J} \begin{bmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{bmatrix} = x \begin{bmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{bmatrix}.$$

- What can be said about

$$\mu(x;t) \leftrightsquigarrow J(t),$$

if the measure is time dependent?

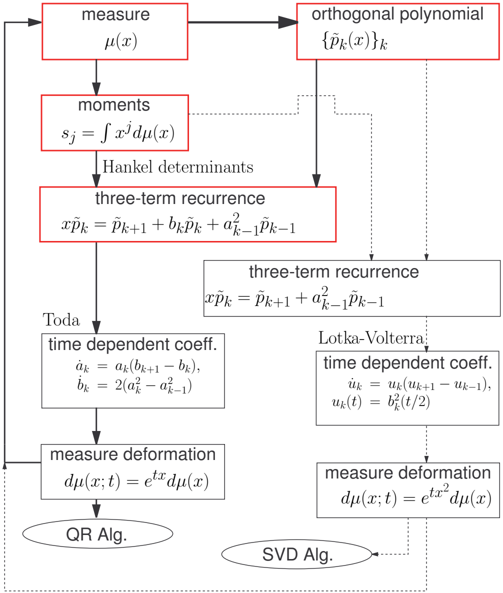
Semi-infinite Toda Lattice

- If $J(t)$ follows the Toda flow, that is, if

$$\begin{aligned} \frac{da_k}{dt} &= a_k(b_{k+1} - b_k), \\ \frac{db_k}{dt} &= 2(a_k^2 - a_{k-1}^2), \end{aligned}$$

with $a_{-1} \equiv 0$, then (Moser'75)

$$d\mu(x;t) := e^{tx} d\mu(x;0).$$


measure
$\mu(x)$
orthogonal polynomial
$\{\tilde{p}_k(x)\}_k$
moments
$s_j = \int x^j d\mu(x)$
Hankel determinants
three-term recurrence
$x\tilde{p}_k = \tilde{p}_{k+1} + b_k\tilde{p}_k + a_{k-1}^2\tilde{p}_{k-1}$
three-term recurrence
$x\tilde{p}_k = \tilde{p}_{k+1} + a_{k-1}^2\tilde{p}_{k-1}$
Toda
time dependent coeff.
$\dot{a}_k = a_k(b_{k+1} - b_k),$
$\dot{b}_k = 2(a_k^2 - a_{k-1}^2)$
Lotka-Volterra
time dependent coeff.
$\dot{u}_k = u_k(u_{k+1} - u_{k-1}),$
$u_k(t) = b_k^2(t/2)$
measure deformation
$d\mu(x;t) = e^{tx}d\mu(x)$
measure deformation
$d\mu(x;t) = e^{tx^2}d\mu(x)$
QR Alg.
SVD Alg.

Finite-Dimensional Eigenvalue Problem

- Truncation.

$$\underbrace{\begin{bmatrix} b_0 & a_0 & 0 & & & \\ a_0 & b_1 & a_1 & 0 & & \\ 0 & a_1 & b_2 & a_2 & 0 & \\ & & \ddots & \ddots & \ddots & \\ & & & & & a_{n-2} \\ & & & 0 & a_{n-2} & b_{n-1} \end{bmatrix}}_{L} \begin{bmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \\ \\ p_{n-1}(x) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \\ a_{n-1}p_n(x) \end{bmatrix} = x \begin{bmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \\ \\ p_{n-1}(x) \end{bmatrix}.$$

- λ is a root of the polynomial $p_n(x)$ if and only if λ is an eigenvalue of the finite-dimensional tridiagonal matrix L.

SVD Flow

- Given a bidiagonal matrix B_0,

$$\frac{dB}{dt} = B\Pi_0(B^\top B) - \Pi_0(BB^\top)B, \quad B(0) = B_0, \tag{1}$$

- $B(t)$ stays bidiagonal for all t.
- The sequence $\{B(\ell)\}_\ell$ by sampling $B(t)$ at integer times corresponds to the iterates produced by the Golub-Kahan SVD algorithm.

Lotka-VOlterra Equation

- Denote

$$B(t) := \operatorname{diag}\left\{ \begin{matrix} & b_2(t) & & \cdots & & b_{2n-2}(t) & \\ b_1(t) & & b_3(t) & & \cdots & & b_{2n-1}(t) \end{matrix} \right\},$$

- Change variables,

$$\begin{aligned} u_{2k-1}(t) &:= b^2_{2k-1}\left(\frac{t}{2}\right), \\ u_{2k}(t) &:= b^2_{2k}\left(\frac{t}{2}\right). \end{aligned}$$

- *Continuous-time finite Lotka-Volterra equation*,

$$\frac{du_k}{dt} = u_k(u_{k+1} - u_{k-1}), \quad k = 1, 2, \ldots, 2n-1, \qquad (2)$$

with $u_0(t) \equiv 0$ and $u_{2n}(t) \equiv 0$.

τ Functions

- Change variables,

$$u_k = \frac{\tau_{k+2}\tau_{k-1}}{\tau_{k+1}\tau_k}.$$

- A compatibility condition,

$$\frac{d\tau_k}{dt}\tau_{k+1} - \tau_k\frac{d\tau_{k+1}}{dt} + \tau_{k-1}\tau_{k+2} = 0.$$

Determinantal Solution

- Starting with $\tau_{-1} \equiv 0$, $\tau_0 \equiv 1$, $\tau_1(t) = 1$ and $\tau_2(t) = \psi(t)$,

$$\begin{aligned} \tau_3 &= \frac{d\psi}{dt}, \\ \tau_4 &= \det \begin{bmatrix} \psi & \psi^{(1)} \\ \psi^{(1)} & \psi^{(2)} \end{bmatrix}, \end{aligned}$$

- In general (Tsujimoto'95),

$$\begin{aligned} \tau_{2k-1} &= \overline{H}_{k-1,1}, \\ \tau_{2k} &= \overline{H}_{k,0}, \end{aligned}$$

where

$$\overline{H}_{k,j}(t) := \det \begin{bmatrix} \psi^{(j)} & \psi^{(j+1)} & \dots & \psi^{(j+k-1)} \\ \psi^{(j+1)} & \psi^{(j+2)} & \dots & \psi^{(j+k)} \\ \vdots & \vdots & & \vdots \\ \psi^{(j+k-1)} & \psi^{(j+k)} & & \psi^{(j+2k-2)} \end{bmatrix}, \quad j = 0 \text{ or } 1,$$

is the determinant of a $k \times k$ Hankel matrix.

SVD Solution

- The general solution to the Lotka-Volterra equation (Tsujimoto, Nakamur&Iwasaki'01)

$$
\begin{aligned}
u_{2k-1}(t) &= \frac{\overline{H}_{k,1}(t)\overline{H}_{k-1,0}(t)}{\overline{H}_{k,0}(t)\overline{H}_{k-1,1}(t)}, \\
u_{2k}(t) &= \frac{\overline{H}_{k+1,0}(t)\overline{H}_{k-1,1}(t)}{\overline{H}_{k,1}(t)\overline{H}_{k,0}(t)}, \quad k = 1, 2, \ldots, n,
\end{aligned}
$$

- Assuming all derivatives of ψ are obtainable from elementary calculus,
  - In principle, all Hankel determinants can be calculated algebraically.
  - All quantities about $u_k(t)$ are now in the analytic form.
  - The SVD flow and, hence, the iterates from the SVD algorithm are representable in closed form.

Conclusion

- Each of the Toda lattice and the Lotka-Volterra equation governs the evolution of a certain class of orthogonal polynomials whose orthogonality is determined by a specific time-dependent measure.
- Since the measure deformation is explicitly known, moments can be calculated which, when properly assembled, lead to the conclusion abstractly, but literally, that the iterates of the *QR* algorithm and the *SVD* algorithm can be expressed in closed-form!
- Hankel determinantal solutions are too complicated to be useful. However, a “smart" integrability-preserving discretization of the Lotka-Volterra equation can yield a new *SVD* algorithm.

A New Chapter

- A key step in the integrable discretization of the Lotka-Volterra equation (2) is a particular Euler-type scheme of the form (Hirota, Tsujimoto&Imai'93),

$$u_k^{[\ell+1]} = u_k^{[\ell]} + \delta \left(u_k^{[\ell]} u_{k+1}^{[\ell]} - u_k^{[\ell+1]} u_{k-1}^{[\ell+1]} \right).$$

  - $u_k^{[\ell]} \approx u_k(\ell\delta)$.
  - Boundary conditions $u_0^{[\ell]} \equiv 0$ and $u_{2n}^{[\ell]} \equiv 0$ for all ℓ.