Fast algorithm for inversion of polynomial–Vandermonde matrices related to (H, m)-quasiseparable matrices

and

Classification of recurrence relations for polynomials via subclasses of (H, m)–quasiseparable matrices

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Joint work with **Vadim Olshevsky, Yuli Eidelman, Israel Gohberg & Eugene Tyrtyshnikov**.

Contents

Introduction

A problem: Inversion/system solving for the classical Vandermonde matrices

- **Definition.** A **Vandermonde matrix** is defined by

$$x = \{x_1, x_2, \dots, x_n\}$$
$$\Downarrow$$
$$V(x) = \begin{bmatrix} 1 & x_1 & {x_1}^2 & \cdots & {x_1}^{n-1} \\ 1 & x_2 & {x_2}^2 & \cdots & {x_2}^{n-1} \\ 1 & x_3 & {x_3}^2 & \cdots & {x_3}^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & {x_n}^2 & \cdots & {x_n}^{n-1} \end{bmatrix}$$

- **Fast algorithms:**
  - **Inversion**: **Traub** (1966) algorithm
  - **System solver**: **Björck-Pereyra** (1970) algorithm
  - **Cost**: $O(n^2)$ vs $O(n^3)$ of Gaussian elimination

Conditioning of Vandermonde matrices

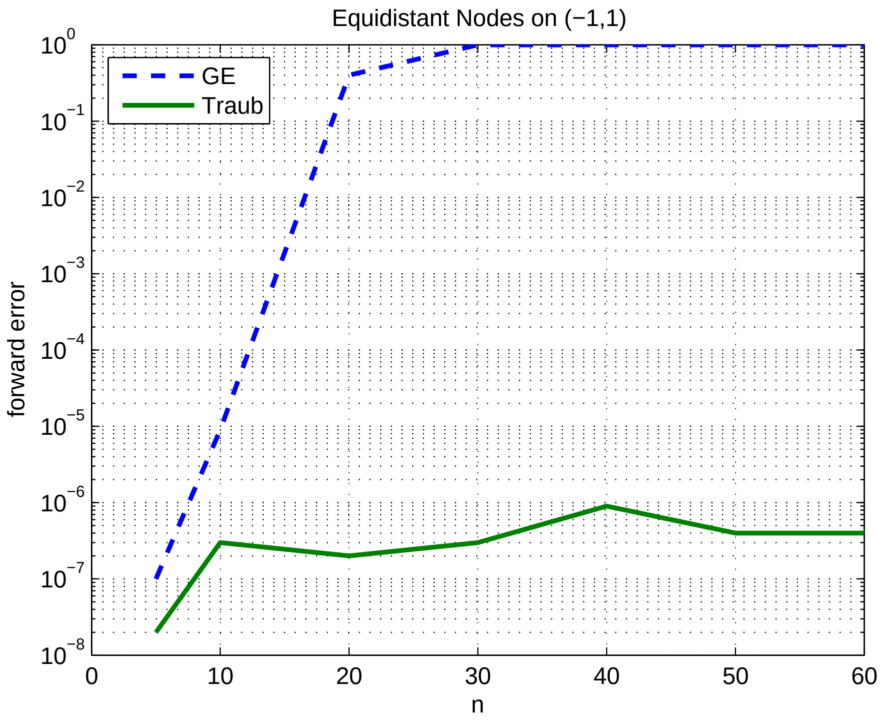
- The condition numbers of Vandermonde matrices **grow exponentially with their size** (Tyrtyshnikov (1994).
- Björck-Pereyra (1970) : "*... some problems, connected with Vandermonde systems, which traditionally have been considered to be too ill-conditioned to be attacked, actually can be solved with good precision*".
- **Björck-Pereyra algorithm.** (**Higham's example**)
  1. Nodes chosen randomly in $(0, 1)$; RHS alternating signs, $\begin{bmatrix} 1 & -1 & 1 & \cdots \end{bmatrix}^T$
  2. Forward error measured by $e = \frac{\|x-\hat{x}\|_2}{\|x\|_2}$

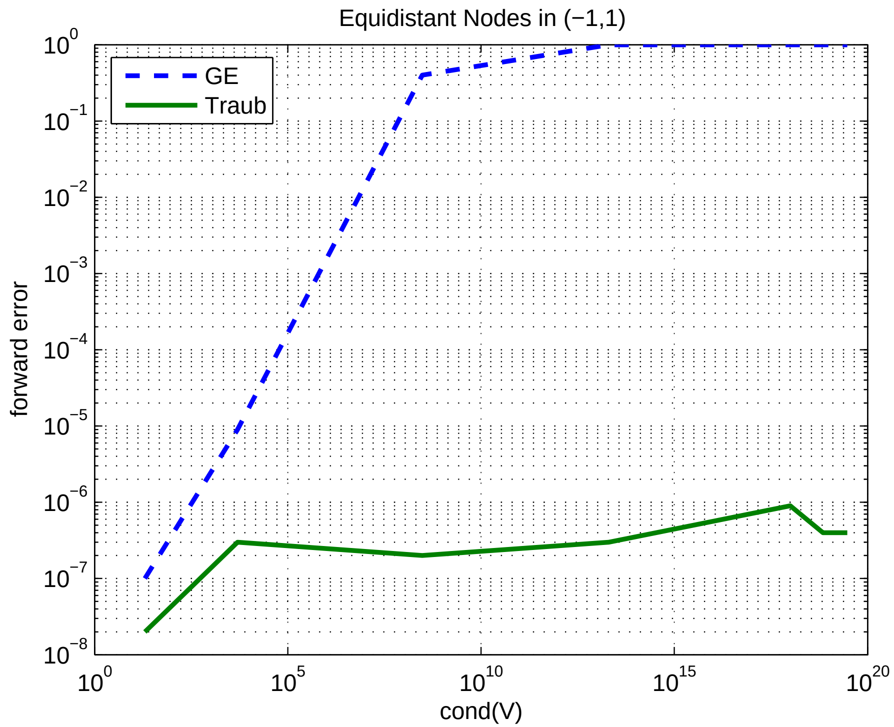
Some Numerical Experiments

- **Traub algorithm.**
  1. Nodes are chosen to be equidistant on $(-1, 1)$.
  2. Forward error measured by $e = \frac{\|A^{-1}-\widehat{A^{-1}}\|_2}{\|A^{-1}\|_2}$

Numerical Experiments

Gaussian Elimination & Traub Algorithm





Polynomial-Vandermonde matrices

- **Definition.** For sets of polynomials and nodes, define a **polynomial-Vandermonde matrix**:

$$x = \{x_1, x_2, \dots, x_n\}$$
$$R = \{r_0(x), r_1(x), \dots, r_{n-1}(x)\}$$
$$\Downarrow$$
$$V_R(x) = \begin{bmatrix} r_0(x_1) & r_1(x_1) & r_2(x_1) & \cdots & r_{n-1}(x_1) \\ r_0(x_2) & r_1(x_2) & r_2(x_2) & \cdots & r_{n-1}(x_2) \\ r_0(x_3) & r_1(x_3) & r_2(x_3) & \cdots & r_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_0(x_n) & r_1(x_n) & r_2(x_n) & \cdots & r_{n-1}(x_n) \end{bmatrix}$$

Fast $\mathcal{O}(n^2)$ algorithms for polynomial-Vandermonde matrices

Previous work

Polynomials $\{r_k(x)\}$	Traub-like algorithm for **inversion**	Björck-Pereyra-like **system solver**
monomials	Traub (1966)	Björck-Pereyra (1970)
Chebyshev	Gohberg-Olshevsky (1994)	Reichel-Opfer (1991)
real-orthogonal	Calvetti-Reichel (1993)	Higham (1988, 90)
Szegö	Olshevsky (2001)	Bella et al (LAA, Jan. 2007)
$(H,1)$–quasi-separable	Bella et al (2007)	Bella et al (2007)

All listed algorithms require only $\mathcal{O}(n^2)$ operations, as opposed to $\mathcal{O}(n^3)$ required by GE.

Fast $\mathcal{O}(n^2)$ algorithms for polynomial-Vandermonde matrices

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(H,m)–quasi-separable	**2007**	**Bella et al (2007)**

All listed algorithms require only $\mathcal{O}(n^2)$ operations, as opposed to $\mathcal{O}(n^3)$ required by GE.

Part 1. (H, m)–quasiseparable matrices and polynomials

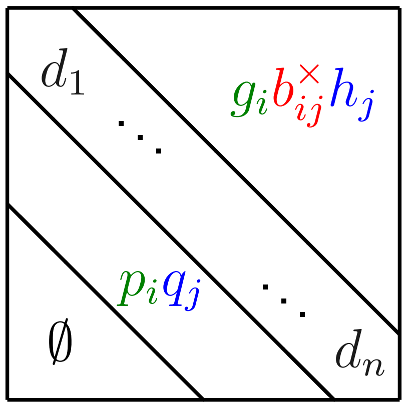
- **Rank Definition.** A matrix C is **(H,m)–quasiseparable** if it is upper Hessenberg and

$$\max \text{Rank} C_{12} = \mathbf{m}$$

where the maxima are taken over all symmetric partitions of the form

$$C = \left[\begin{array}{c|c} * & C_{12} \\ \hline * & * \end{array}\right]$$

- **Generator Definition** The following matrix is **Hessenberg** and order–**m** quasiseparable:



where $\{g_i\}$, $\{b_i\}$, $\{h_i\}$ are matrices of sizes $1 \times r_i$, $r_{i-1} \times r_i$ and $r_{i-1} \times 1$ respectively.

$$\max_{1 \leqslant i \leqslant N-1} r_i = \mathbf{m}$$

The set of (H, m)-quasiseparable matrices contains:

- $(H, 1)$-quasiseparable matrices
  - Tridiagonal matrices

$$\begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

  - Unitary Hessenberg matrices

$$\begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & -\rho_0^*\mu_1\mu_2\mu_3\rho_4 & -\rho_0^*\mu_1\mu_2\mu_3\mu_4\rho_5 \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & -\rho_1^*\mu_2\mu_3\rho_4 & -\rho_1^*\mu_2\mu_3\mu_4\rho_5 \\ 0 & \mu_2 & -\rho_2^*\rho_3 & -\rho_2^*\mu_3\rho_4 & -\rho_2^*\mu_3\mu_4\rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^*\rho_4 & -\rho_3^*\mu_4\rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^*\rho_5 \end{bmatrix}$$

- Matrices not covered by $(H, 1)$-case!!!

It is easy to find an example of (H, m)–qs polynomials and matrices!

If polynomials satisfy simplest l-term recurrence relations

$$r_k(x) = (\alpha_k x - a_{k-1,k}) \cdot r_{k-1}(x) - a_{k-2,k} \cdot r_{k-2}(x) - \ldots - a_{k-(l-1),k} \cdot r_{k-(l-1)}(x)$$

then their confederate matrices

$$A = \begin{bmatrix} \frac{a_{0,1}}{\alpha_1} & \cdots & \frac{a_{0,l-1}}{\alpha_l - 1} & 0 & \cdots & 0 \\ \frac{1}{\alpha_1} & \frac{a_{1,2}}{\alpha_2} & \cdots & \frac{a_{1,l}}{\alpha_l} & \ddots & \vdots \\ 0 & \frac{1}{\alpha_2} & & & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \frac{a_{n-(l-1),n}}{\alpha_n} \\ \vdots & & \ddots & \frac{1}{\alpha_{n-2}} & & \vdots \\ 0 & \cdots & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} \end{bmatrix}$$

are $(1, l-2)$-banded, i.e., they have only one nonzero subdiagonal and $l-2$ nonzero superdiagonals. Clearly A is an (H, m)–quasiseparable matrix for $m = l - 2$ by definition.

Two–term recurrence relations for polynomials

- Let C be an (H, m)–quasiseparable matrix. Then the system of polynomials

$$\mathbf{r_k}(x) = \frac{1}{c_{2,1}c_{3,2}\cdots c_{n,n-1}} \det(xI - C_{k\times k})$$

associated with C satisfy the recurrence relation

$$\begin{bmatrix} F_k(x) \\ \hline \mathbf{r_k}(x) \end{bmatrix} = \frac{1}{p_{k+1}q_k} \left[\begin{array}{c|c} p_k q_k \, {b_k}^T & -q_k \, {g_k}^T \\ \hline p_k \, {h_k}^T & x - d_k \end{array}\right] \begin{bmatrix} F_{k-1}(x) \\ \hline \mathbf{r_{k-1}}(x) \end{bmatrix}$$

where $F_k(x)$ - the vector of auxiliary polynomials.

Part 2. Fast Traub-like algorithm for (H, m)–quasiseparable Vandermonde matrices

(H, m)–quasiseparable-Vandermonde matrices

- **Definition.** An (H, m)**–quasiseparable-Vandermonde matrix** is of the form

$$V_R = \begin{bmatrix} r_0(x_1) & r_1(x_1) & r_2(x_1) & \cdots & r_{n-1}(x_1) \\ r_0(x_2) & r_1(x_2) & r_2(x_2) & \cdots & r_{n-1}(x_2) \\ r_0(x_3) & r_1(x_3) & r_2(x_3) & \cdots & r_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_0(x_n) & r_1(x_n) & r_2(x_n) & \cdots & r_{n-1}(x_n) \end{bmatrix}$$

  where the polynomials $r_k(x)$ defined by

$$r_k(x) = \frac{1}{c_{2,1}c_{3,2}\cdots c_{n,n-1}} \det(xI - C_{k\times k})$$

  correspond to an (H, m)**–quasiseparable** matrix C.

- The class of (H, m)**–quasiseparable** polynomials contains as subclasses the classes of **real–orthogonal polynomials** and **Szegö polynomials**.

Traub–like algorithm for (H,m)–quasiseparable-Vandermonde matrices

- **Traub–like algorithm**.

  Based on the formula

$$V_R^{-1} = \begin{bmatrix} r_0(x_1) & r_1(x_1) & \cdots & r_{n-1}(x_1) \\ r_0(x_2) & r_1(x_2) & \cdots & r_{n-1}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ r_0(x_n) & r_1(x_n) & \cdots & r_{n-1}(x_n) \end{bmatrix}^{-1} = \tilde{I} \cdot V_{\hat{R}}^{T} \cdot \text{diag}(c_1, c_2, \ldots, c_n)$$

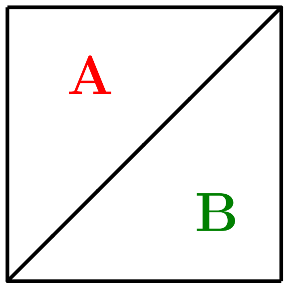
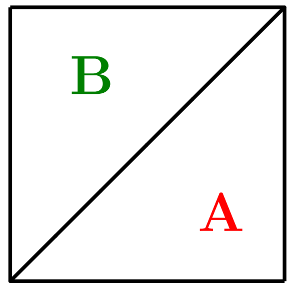
  where $\tilde{I}$ is the antidiagonal matrix, $c_k = \prod_{\substack{j=1 \\ j\neq k}}^{n} (x_j - x_k)^{-1}$

- $\hat{R}$ is the system of **Horner–like** polynomials corresponding to the polynomial system R. (When R is the monomial basis, this is the classical Traub (1966))

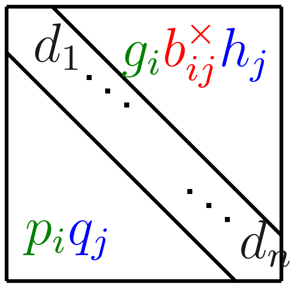
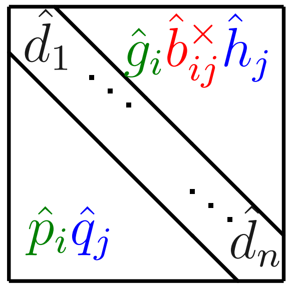
- **How do we evaluate the polynomials $\hat{r}_k$ at the nodes?**

Via Pertransposition!

- **Pertransposition** of the related matrix provides a relation between the matrix corresponding to a system of polynomials R and the matrix corresponding to the **associated** polynomials $\hat{R}$.

		
C	$\Rightarrow$ **pertransposition**	$\tilde{I} \cdot C^T \cdot \tilde{I}$
$\{r_0(x), r_1(x), \ldots, r_{n-1}(x)\}$		$\{\hat{r}_0(x), \hat{r}_1(x), \ldots, \hat{r}_{n-1}(x)\}$
Polynomial System		Associated polynomials

Associated (Horner-like) polynomials

$C_R(P) =$ 		$C_{\widehat{R}}(P) =$ 
Q/S-Hessenberg Matrix	$\Rightarrow$ **pertransposition**	Q/S-Hessenberg Matrix
$\{r_0(x), r_1(x), \ldots, r_{n-1}(x)\}$		$\{\hat{r}_0(x), \hat{r}_1(x), \ldots, \hat{r}_{n-1}(x)\}$
Polynomial System		Associated polynomials

Relation of generators

- **Before pertransposition: polynomials** $R = \{r_0(x), \dots, r_{n-1}(x)\}$

$$C_R(P) = \begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ 0 & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ 0 & 0 & p_4q_3 & d_4 & g_4h_5 \\ 0 & 0 & 0 & p_5q_4 & d_5 \end{bmatrix} - \begin{bmatrix} \mathbf{P_0} \\ \mathbf{P_1} \\ \mathbf{P_2} \\ \mathbf{P_3} \\ \mathbf{P_4} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{\mathbf{P_5}} \end{bmatrix}$$

- **After pertransposition: associated polynomials** $\widehat{R} = \{\widehat{r}_0(x), \dots, \widehat{r}_{n-1}(x)\}$

$$C_{\widehat{R}}(P) = \begin{bmatrix} d_5 & g_4h_5 & g_3b_4h_5 & g_2b_3b_4h_5 & g_1b_2b_3b_4h_5 \\ p_5q_4 & d_4 & g_3h_4 & g_2b_3h_4 & g_1b_2b_3h_4 \\ 0 & p_4q_3 & d_3 & g_2h_3 & g_1b_2h_3 \\ 0 & 0 & p_3q_2 & d_2 & g_1h_2 \\ 0 & 0 & 0 & p_2q_1 & d_1 \end{bmatrix} - \begin{bmatrix} \frac{1}{\mathbf{P_5}} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \mathbf{P_4} & \mathbf{P_3} & \mathbf{P_2} & \mathbf{P_1} & \mathbf{P_0} \end{bmatrix}$$

Theorem: recurrence relations for the associated polynomials

$$\left[\begin{array}{c} \widehat{F}_k(x) \\ \hline \widehat{r}_k(x) \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline P_n \end{array}\right],$$

$$\left[\begin{array}{c} \widehat{F}_k(x) \\ \hline \widehat{r}_k(x) \end{array}\right] = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k} \left[\begin{array}{c|c} \widehat{p}_k\widehat{q}_k \, \widehat{b}_k{}^T & -\widehat{q}_k \, \widehat{g}_k{}^T \\ \hline \widehat{p}_k \, \widehat{h}_k{}^T & x - \widehat{d}_k \end{array}\right] \left[\begin{array}{c} \widehat{F}_{k-1}(x) \\ \hline \widehat{r}_k(x) \end{array}\right] + \frac{1}{\widehat{p}_{k+1}\widehat{q}_k} \left[\begin{array}{c} 0 \\ \hline P_{n-k} \end{array}\right]$$

What are the coefficients P_k?

- The difference between the recurrence relations for the original polynomials and those for the Horner-like polynomials is the presence of the coefficients P_k.
- P_k is the coefficient of $r_k(x)$ in the decomposition of the **master polynomial**

$$P(x) = \prod_{i=1}^{n}(x - x_i)$$

into the $\{r_k\}$ basis:

$$\prod_{i=1}^{n}(x - x_i) = P_0 r_0(x) + P_1 r_1(x) + \cdots + P_n r_n(x)$$

- These coefficients can be computed in $\mathcal{O}(n^2)$ arithmetic operations.

Complexity of the (H, m)-Traub-like algorithm

- Each Horner-like polynomial can be evaluated at all of the nodes in $\mathcal{O}(n)$ operations.
- The coefficients P_k can be computed in $\mathcal{O}(n^2)$ operations.
- The total cost of the algorithm is $\mathcal{O}(n^2)$ operations. Comparing this to the complexity of Gaussian elimination, $\mathcal{O}(n^3)$, we have the algorithm is **FAST!**

Part 3. Numerical experiments

Description

- We compare the **forward error** of the inverse A_s^{-1} from C++ code in single precision via

$$e = \frac{\|A_d^{-1} - A_s^{-1}\|_2}{\|A_d^{-1}\|_2},$$

with A_d^{-1}, the “exact” solution computed in double precision.

- **GE** - Gaussian elimination via Lapack subroutine SGESV.
- For all additional matrix operations and algorithms such as matrix multiplication and SVD we used Lapack subroutines.
- **Traub**(H, m) - Traub-like algorithm with nodes ordered via the **Leja ordering**. (Reichel, Higham)

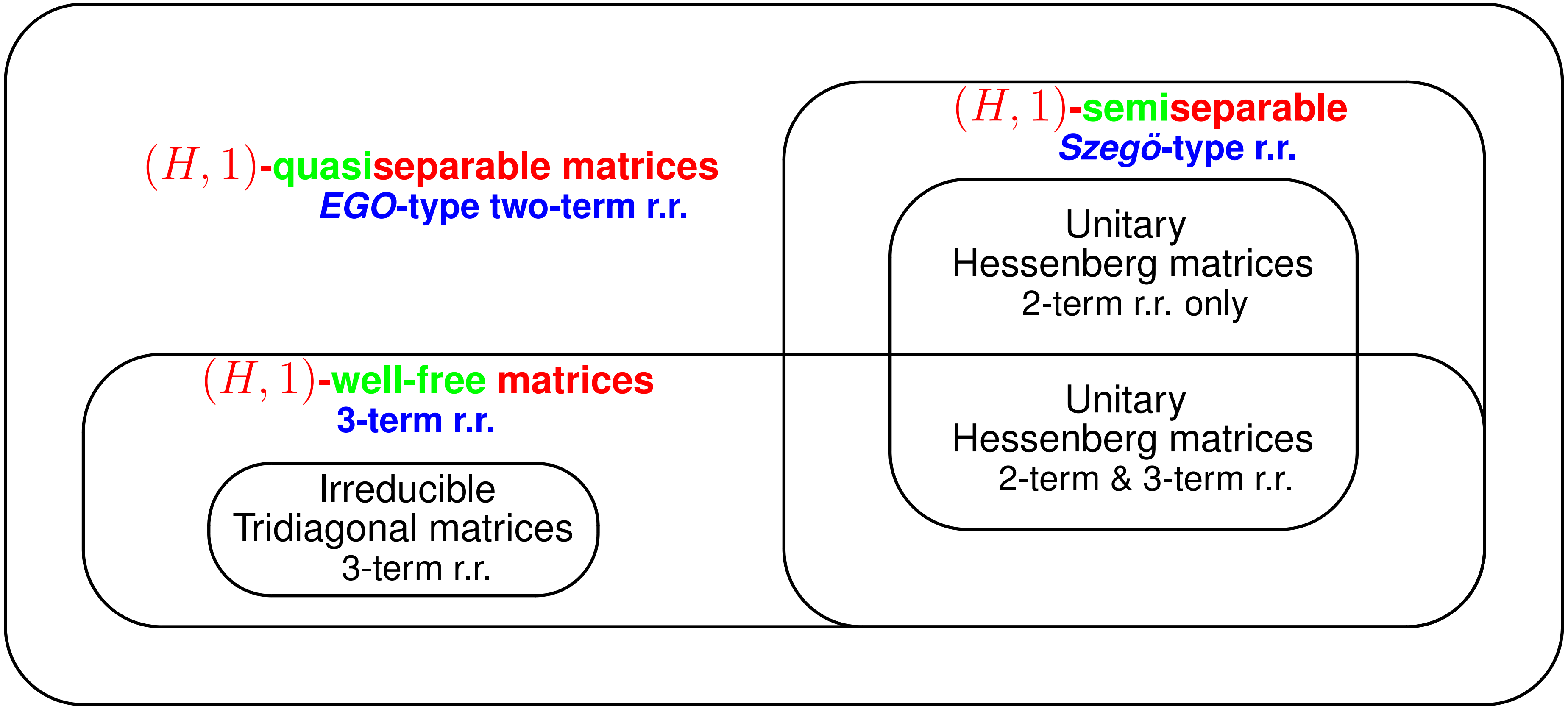
Random generators on (0,1) & equidistant nodes on (-1,1)

m	n	cond(V)	GE	Traub(H,m)
1	10	1.8e+007	1.5e-006	1.4e-006
	20	2.2e+020	5.9e-001	5.0e-007
	30	7.2e+029	1.2e+000	2.6e-006
	40	2.1e+034	1.0e+000	1.6e-005
	50	1.5e+039	1.0e+000	3.9e-007
2	10	5.6e+007	4.6e-005	5.4e-007
	20	1.6e+019	2.6e+000	1.9e-007
	30	3.4e+025	9.2e-001	2.7e-006
	40	3.3e+033	1.0e+000	1.9e-006
	50	3.7e+038	1.0e+000	7.6e-001
3	20	1.0e+021	5.6e-002	2.3e-006
	30	2.9e+029	1.0e+000	2.0e-006
	40	4.1e+029	1.0e+000	1.1e-003
	50	2.6e+041	1.0e+000	4.5e-004

m	n	cond(V)	GE	Traub(H,m)
4	20	6.1e+020	2.8e+000	5.5e-006
	30	7.5e+026	1.0e+000	1.5e-006
	40	4.5e+028	1.0e+000	3.5e-007
	50	2.0e+037	1.0e+000	3.8e-001
5	30	5.0e+024	1.0e+000	1.4e-006
	40	1.2e+031	1.0e+000	7.2e-007
	50	1.7e+037	1.0e+000	5.7e-007
6	30	2.5e+026	1.0e+000	5.1e-007
	40	3.5e+032	1.0e+000	3.3e-006
	50	8.0e+038	1.0e+000	9.2e-005
7	40	6.0e+027	1.0e+000	1.7e-004
	50	1.7e+038	1.0e+000	9.3e-007
8	40	7.8e+031	1.0e+000	5.0e-007
	50	7.5e+036	1.0e+000	4.7e-007

Part 4. Classification of recurrence relations for polynomials via subclasses of (H, m)–quasiseparable matrices

Subclasses of $(H, \mathbf{1})$-qs matrices and polynomials



Quasiseparable matrices and polynomials

$(H, 1)$-**quasiseparable matrices**

$$\Updownarrow$$

***EGO*-type recurrence relations**

$$\begin{bmatrix} F_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k x + \theta_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}$$

Quasiseparable matrices and polynomials

(H, m)-**quasiseparable matrices**

$$\Updownarrow$$

Block *EGO*-type recurrence relations

$$\left[\begin{array}{c} F_0(x) \\ \hline r_0(x) \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 1 \end{array}\right], \quad \left[\begin{array}{c} F_k(x) \\ \hline r_k(x) \end{array}\right] = \left[\begin{array}{c|c} \alpha_k & \beta_k \\ \hline \gamma_k & \delta_k x + \theta_k \end{array}\right] \left[\begin{array}{c} F_{k-1}(x) \\ \hline r_{k-1}(x) \end{array}\right]$$

Subclasses of $(H, \mathbf{m})$-qs matrices and polynomials

(H, m)-quasiseparable matrices
Block *EGO*-type two-term r.r.

Semiseparable matrices and polynomials

$(H, 1)$-semiseparable matrices

$$\Updownarrow$$

***Szegö*-type recurrence relations**

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k x + \theta_k) r_{k-1}(x) \end{bmatrix}$$

Semiseparable matrices and polynomials

(H, m)-semiseparable matrices

$$\Updownarrow$$

Block *Szegö*-type recurrence relations

$$\left[\begin{array}{c} G_0(x) \\ \hline r_0(x) \end{array}\right] = \left[\begin{array}{c} 1 \\ \vdots \\ 1 \\ \hline 1 \end{array}\right], \quad \left[\begin{array}{c} G_k(x) \\ \hline r_k(x) \end{array}\right] = \left[\begin{array}{c|c} \alpha_k & \beta_k \\ \hline \gamma_k & 1 \end{array}\right] \left[\begin{array}{c} G_{k-1}(x) \\ \hline (\delta_k x + \theta_k) \times \\ r_{k-1}(x) \end{array}\right]$$

Subclasses of $(H, \mathbf{m})$-qs matrices and polynomials

(H, m)-quasiseparable matrices
Block *EGO*-type two-term r.r.

(H, m)-semiseparable
Block *Szegö*-type r.r.

Well-free matrices and polynomials

Polynomials satisfying three-term recurrence relations

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - (\boxed{\beta_k} x + \boxed{\gamma_k}) r_{k-2}(x)$$

contain both:

- Real-orthogonal polynomials: $\boxed{\beta_k} = 0$

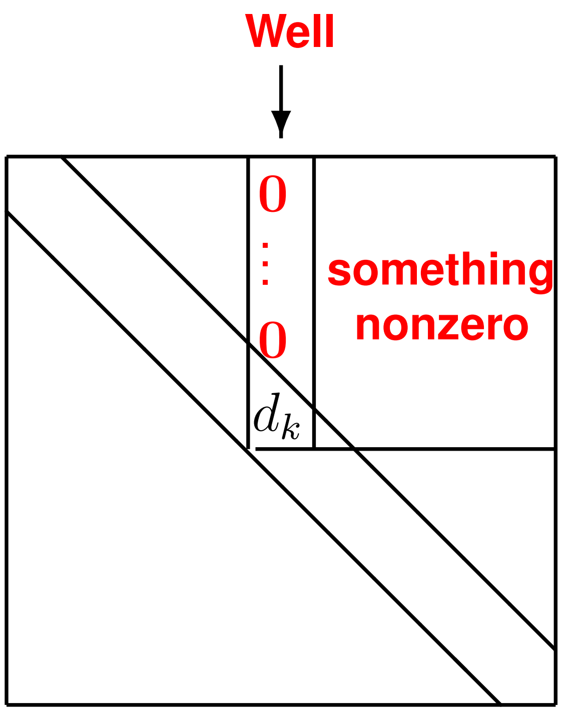
$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - \boxed{\gamma_k} r_{k-2}(x)$$

- Szegö polynomials (orthogonal on the unit circle): $\boxed{\gamma_k} = 0$

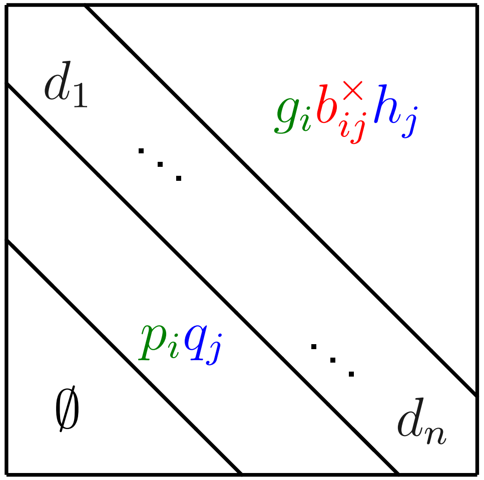
$$r_k(x) = \left(\frac{1}{\mu_k} x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k} \right) r_{k-1}(x) - \left(\boxed{\frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k}} \cdot x \right) r_{k-2}(x)$$

Well-free matrices and polynomials

Well free matrix in the $(H, 1)$-case is defined by



Well-free $\Leftrightarrow h_j \neq 0$



- Irreducible tridiagonal are well-free.
- Unitary Hessenberg ($\rho_k \neq 0$) are well-free.

Well-free matrices and polynomials

In order to generalize three-term recurrence relations

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - (\beta_k x + \gamma_k) r_{k-2}(x)$$

have a look at the simplest l-term ones

$$r_k(x) = (\alpha_k x - a_{k-1,k}) \cdot r_{k-1}(x) - a_{k-2,k} \cdot r_{k-2}(x) - \ldots - a_{k-(l-1),k} \cdot r_{k-(l-1)}(x)$$

and . . .

Combine!

Well-free matrices and polynomials

(H, m)-well-free matrices ($m = l - 2$)

$$\Updownarrow$$

General l-term recurrence relations

$$r_0(x) = 1, \quad r_k(x) = \sum_{i=1}^{k} (\delta_{ik} x + \varepsilon_{ik}) r_{i-1}(x), \quad k = 1, 2, \ldots, l-2$$

$$r_k(x) = \sum_{i=k-l+2}^{k} (\delta_{ik} x + \varepsilon_{ik}) r_{i-1}(x), \quad k = l-1, l, \ldots, n$$

But what are (H, m)-well-free matrices?

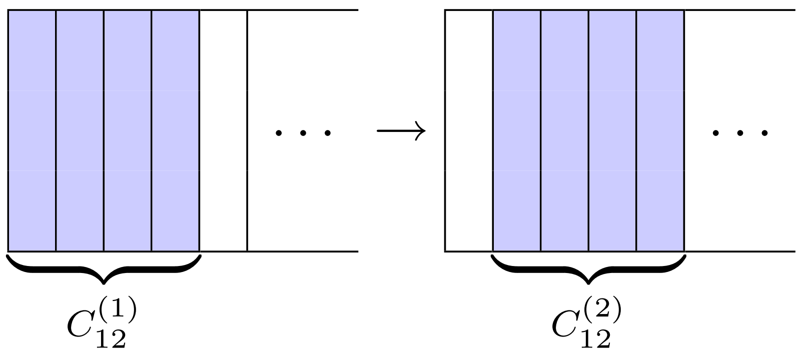
Well-free matrices and polynomials

(H, m)-well-free matrices characterization

Let C be an (H, m)**-quasiseparable matrix**. Have a look at all it symmetric partitions C_{12} of the form

$$C = \left[\begin{array}{c|c} * & C_{12} \\ \hline * & * \end{array}\right]$$

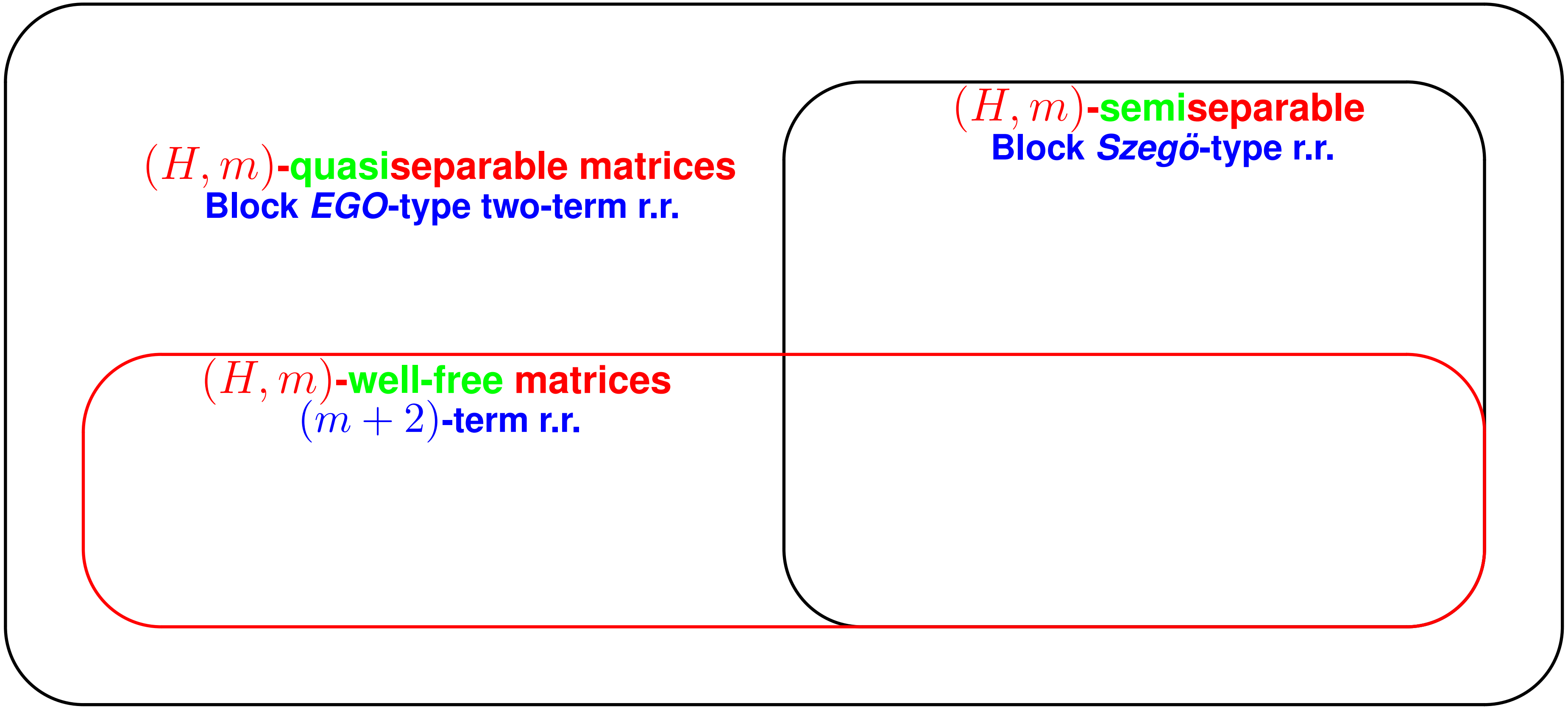
and a sliding window of m columns in C_{12}:



then C is (H, m)**-well-free** if

$$\text{Rank}\,\mathbf{C_{12}^{(i+1)}} \leqslant \text{Rank}\,\mathbf{C_{12}^{(i)}}$$

Subclasses of $(H, \mathbf{m})$-qs matrices and polynomials



Final Venn diagram of subclasses of (H, m)-quasiseparable matrices and polynomials