

Quasiseparable matrices and polynomials. Fast and accurate algorithms.

Tom Bella

Department of Mathematics
University of Connecticut
Storrs, CT 06269 USA

Joint work with:

Yuli Eidelman, Israel Gohberg, Israel Koltracht,
Vadim Olshevsky & Eugene Tyrtyshnikov

Structured Matrices

- ▶ Many problems in signal processing, coding theory, system theory, orthogonal polynomials, passive interpolation, etc. can be reduced to problems in terms of matrices.
- ▶ Often these resulting matrices have some structure inherited from the original problem, e.g. Vandermonde, Hankel, Toeplitz, tridiagonal, unitary Hessenberg, etc.
- ▶ By exploiting this structure, it is often possible to attain **more accurate results** using **fewer operations** than standard, structure-ignoring methods.

Examples of Structured Matrices

Moment Matrices

► **Hankel matrices.** Defined by $\mathcal{O}(n)$ parameters $\{h_k\}$.

$$H = \begin{bmatrix} h_{k+j} \end{bmatrix} = \begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{n-1} \\ h_1 & h_2 & \ddots & \ddots & \vdots \\ h_2 & \ddots & \ddots & \ddots & h_{2n-3} \\ \vdots & \ddots & & h_{2n-3} & h_{2n-2} \\ h_{n-1} & \cdots & h_{2n-3} & h_{2n-2} & h_{2n-1} \end{bmatrix}$$

► **Toeplitz matrices.** Defined by $\mathcal{O}(n)$ parameters $\{c_k\}$.

$$C = \begin{bmatrix} c_{k-j} \end{bmatrix} = \begin{bmatrix} c_0 & c_{-1} & \cdots & \cdots & c_{-n+1} \\ c_1 & c_0 & c_{-1} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & c_0 & c_{-1} \\ c_{n-1} & \cdots & \cdots & c_1 & c_0 \end{bmatrix}$$

Examples of Structured Matrices

Moment Matrices

$$M = [\langle x^k, x^j \rangle] = \begin{bmatrix} \langle 1, 1 \rangle & \langle 1, x \rangle & \langle 1, x^2 \rangle & \dots & \langle 1, x^n \rangle \\ \langle x, 1 \rangle & \langle x, x \rangle & \langle x, x^2 \rangle & \dots & \langle x, x^n \rangle \\ \langle x^2, 1 \rangle & \langle x^2, x \rangle & \langle x^2, x^2 \rangle & \dots & \langle x^2, x^n \rangle \\ \vdots & \vdots & \vdots & & \vdots \\ \langle x^n, 1 \rangle & \langle x^n, x \rangle & \langle x^n, x^2 \rangle & \dots & \langle x^n, x^n \rangle \end{bmatrix}$$

▣▣▣ Real line.

$$\langle p(x), q(x) \rangle = \int_a^b p(x)q(x)w^2(x)dx, \quad \Rightarrow \quad \langle x^k, x^j \rangle = \int_a^b x^{(k+j)}w^2(x)dx,$$

and M is **Hankel**.

▣▣▣ Unit circle.

$$\langle p(x), q(x) \rangle = \int_{-\pi}^{\pi} p(e^{i\theta}) \cdot \overline{q(e^{i\theta})} w^2(\theta) d\theta \Rightarrow \langle x^k, x^j \rangle = \int_0^{2\pi} x^{(k-j)} w^2(\theta) d\theta,$$

and M is **Toeplitz**.

Examples of Structured Matrices

Recurrent Matrices

▣ **Tridiagonal matrices.** Defined by $\mathcal{O}(n)$ parameters.

$$T = \begin{bmatrix} \delta_1 & \gamma_2 & 0 & \cdots & 0 \\ \gamma_2 & \delta_2 & \gamma_3 & \ddots & \vdots \\ 0 & \gamma_3 & \delta_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \gamma_n \\ 0 & \cdots & 0 & \gamma_n & \delta_n \end{bmatrix}$$

▣ **Unitary Hessenberg matrices.** Defined by $\mathcal{O}(n)$ parameters.

$$U = \begin{bmatrix} -\rho_1 \rho_0^* & -\rho_2 \mu_1 \rho_0^* & \cdots & -\rho_n \mu_{n-1} \cdots \mu_1 \rho_0^* \\ \mu_1 & -\rho_2 \rho_1^* & \cdots & -\rho_n \mu_{n-1} \cdots \mu_2 \rho_1^* \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & -\rho_n \mu_{n-1} \rho_{n-2}^* \\ 0 & \cdots & \mu_{n-1} & -\rho_n \rho_{n-1}^* \end{bmatrix}$$

Orthogonal polynomials related to structured matrices

Moment matrix	Recurrent matrix	Polynomials $r_k(x)$
Hankel matrices	tridiagonal matrices	Real-orthogonal polynomials
Toeplitz matrices	unitary Hessenberg matrices	Szegő polynomials

Orthogonal polynomials related to structured matrices

Moment matrix	Recurrent matrix	Polynomials $r_k(x)$
Hankel matrices	tridiagonal matrices	Real-orthogonal polynomials
Toeplitz matrices	unitary Hessenberg matrices	Szegő polynomials

Related to tridiagonal matrices via $\mathbf{r}_k(x) = \det(xI - A)_{(k \times k)}$ where

$$A = \begin{bmatrix} \frac{\delta_1}{\alpha_1} & \frac{\gamma_2}{\alpha_2} & 0 & \cdots & 0 \\ \frac{1}{\alpha_1} & \frac{\delta_2}{\alpha_2} & \ddots & \ddots & \vdots \\ 0 & \frac{1}{\alpha_2} & \ddots & \frac{\gamma_{n-1}}{\alpha_{n-1}} & 0 \\ \vdots & & \ddots & \frac{\delta_{n-1}}{\alpha_{n-1}} & \frac{\gamma_n}{\alpha_n} \\ 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{\delta_n}{\alpha_n} \end{bmatrix}$$

Three-term recurrence relations

$$\mathbf{r}_k(x) = (\alpha_k \mathbf{x} - \delta_k) \mathbf{r}_{k-1}(x) - \gamma_k \mathbf{r}_{k-2}(x)$$

Orthogonal polynomials related to structured matrices

Moment matrix	Recurrent matrix	Polynomials $r_k(x)$
Hankel matrices	tridiagonal matrices	Real-orthogonal polynomials
Toeplitz matrices	unitary Hessenberg matrices	Szegő polynomials

Related to unitary Hessenberg matrices via $\mathbf{r}_k(x) = \det(xI - A)_{(k \times k)}$ where

$$A = \begin{bmatrix} -\rho_1\rho_0^* & -\rho_2\mu_1\rho_0^* & -\rho_3\mu_2\mu_1\rho_0^* & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_1\rho_0^* & -\rho_n\mu_{n-1}\cdots\mu_1\rho_0^* \\ \mu_1 & -\rho_2\rho_1^* & -\rho_3\mu_2\rho_1^* & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_2\rho_1^* & -\rho_n\mu_{n-1}\cdots\mu_2\rho_1^* \\ 0 & \mu_2 & -\rho_3\rho_2^* & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_3\rho_2^* & -\rho_n\mu_{n-1}\cdots\mu_3\rho_2^* \\ \vdots & \ddots & \mu_3 & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & -\rho_{n-1}\rho_{n-2}^* & -\rho_n\mu_{n-1}\rho_{n-2}^* \\ 0 & \cdots & \cdots & 0 & \mu_{n-1} & -\rho_n\rho_{n-1}^* \end{bmatrix}$$

Orthogonal polynomials related to structured matrices

Moment matrix	Recurrent matrix	Polynomials $r_k(x)$
Hankel matrices	tridiagonal matrices	Real-orthogonal polynomials
Toeplitz matrices	unitary Hessenberg matrices	Szegő polynomials

Two-term recurrence relations

$$\begin{bmatrix} G_{k+1}(x) \\ \mathbf{r}_{k+1}(x) \end{bmatrix} = \frac{1}{\mu_{k+1}} \begin{bmatrix} 1 & -\rho_{k+1}^* \\ -\rho_{k+1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{x} \end{bmatrix} \begin{bmatrix} G_k(x) \\ \mathbf{r}_k(x) \end{bmatrix}.$$

Three-term recurrence relations

$$\mathbf{r}_k(x) = \left(\frac{1}{\mu_k} x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k} \right) \mathbf{r}_{k-1}(x) - \left(\frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x \right) \mathbf{r}_{k-2}(x)$$

Generalizations of these structures

Matrix class	Generalized class
Hankel matrices Toeplitz matrices	matrices with displacement structure
tridiagonal matrices unitary Hessenberg matrices	????????????????????

Generalizations of these structures

Matrix class	Generalized class
Hankel matrices Toeplitz matrices	matrices with displacement structure
tridiagonal matrices unitary Hessenberg matrices	quasiseparable matrices

Vandermonde matrices and algorithms

►►► **Definition.** For a set of nodes, a **Vandermonde matrix** is defined by

$$\begin{aligned} x &= \{x_1, x_2, \dots, x_n\} \\ &\Downarrow \\ V(x) &= \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix} \end{aligned}$$

Algorithms available for Vandermonde matrices

►►► Björck-Pereyra algorithm

►►► Traub algorithm

Fast Björck-Pereyra algorithm

► The **Björck-Pereyra algorithm** (1970) is based on the formula

$$V(x)^{-1} = U_1^{-1} \cdots U_{n-1}^{-1} L_{n-1}^{-1} \cdots L_1^{-1},$$

with

$$U_k^{-1} = \left[\begin{array}{c|ccc} I_{k-1} & & & \\ \hline & 1 & -x_k & \\ & & 1 & \ddots \\ & & & \ddots & -x_k \\ & & & & 1 \end{array} \right]$$

$$L_k^{-1} = \left[\begin{array}{c|ccc} I_{k-1} & & & \\ \hline & 1 & & \\ & & \frac{1}{x_{k+1}-x_k} & \\ & & & \ddots \\ & & & & \frac{1}{x_n-x_k} \end{array} \right] \left[\begin{array}{c|ccc} I_{k-1} & & & \\ \hline & 1 & & \\ & -1 & 1 & \\ & \vdots & & \ddots \\ & -1 & & & 1 \end{array} \right]$$

Fast Björck-Pereyra algorithm

- ▶▶▶ The solution a of the linear system

$$V(x)a = f$$

is computed by Björck-Pereyra as

$$a = V(x)^{-1}f = U_1^{-1} \dots U_{n-1}^{-1} L_{n-1}^{-1} \dots L_1^{-1} f$$

- ▶▶▶ Each matrix in the factorization is **sparse**, and so each matrix-vector product can be computed in $\mathcal{O}(n)$ operations.
- ▶▶▶ **Björck-Pereyra** requires only $\mathcal{O}(n^2)$ arithmetic operations vs $\mathcal{O}(n^3)$ of Gaussian elimination.

Fast Traub algorithm

► The **Traub algorithm** (1966) is based on the formula

$$V(x)^{-1} = \tilde{I} \cdot \begin{bmatrix} r_0(x_1) & r_1(x_1) & r_2(x_1) & \cdots & r_{n-1}(x_1) \\ r_0(x_2) & r_1(x_2) & r_2(x_2) & \cdots & r_{n-1}(x_2) \\ r_0(x_3) & r_1(x_3) & r_2(x_3) & \cdots & r_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_0(x_n) & r_1(x_n) & r_2(x_n) & \cdots & r_{n-1}(x_n) \end{bmatrix}^T \cdot \text{diag}(c_1, c_2, \dots, c_n)$$

where \tilde{I} is the antidiagonal matrix, $c_k = \prod_{\substack{j=1 \\ j \neq k}}^n (x_j - x_k)^{-1}$

► The polynomials $\{r_0(x), \dots, r_{n-1}(x)\}$ are the **Horner (associated) polynomials**, and satisfy **two-term recurrence relations**

$$r_0(x) = P_n, \quad r_k(x) = xr_{k-1}(x) + P_{n-k}$$

► **Fast:** requires only $\mathcal{O}(n^2)$ arithmetic operations vs $\mathcal{O}(n^3)$ of Gaussian elimination.

Conditioning of Vandermonde matrices

- ▶▶▶ The condition numbers of Vandermonde matrices **grow exponentially with their size** (Tyrtysnikov (1994)).
- ▶▶▶ Björck-Pereyra (1970) : “... *some problems, connected with Vandermonde systems, which traditionally have been considered to be too ill-conditioned to be attacked, actually can be solved with good precision*”.

Some Numerical Experiments

▣▣▣▣ Björck-Pereyra algorithm. (Higham's example)

1. Nodes chosen randomly in $(0, 1)$.
2. RHS alternating signs, $\begin{bmatrix} 1 & -1 & 1 & \cdots \end{bmatrix}^T$
3. Forward error measured by

$$e = \frac{\|x - \hat{x}\|_2}{\|x\|_2}$$

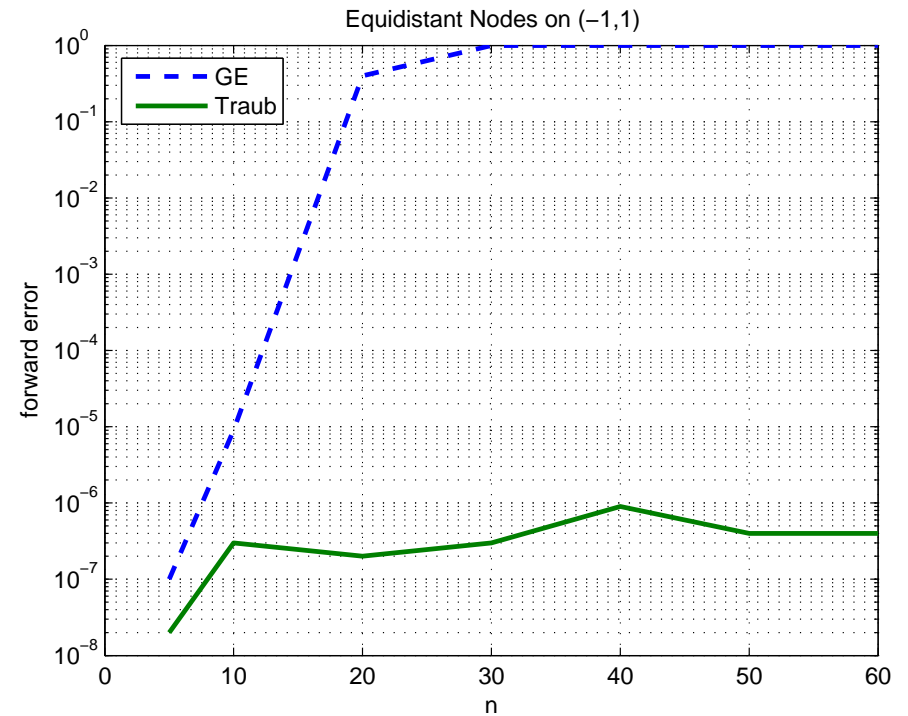
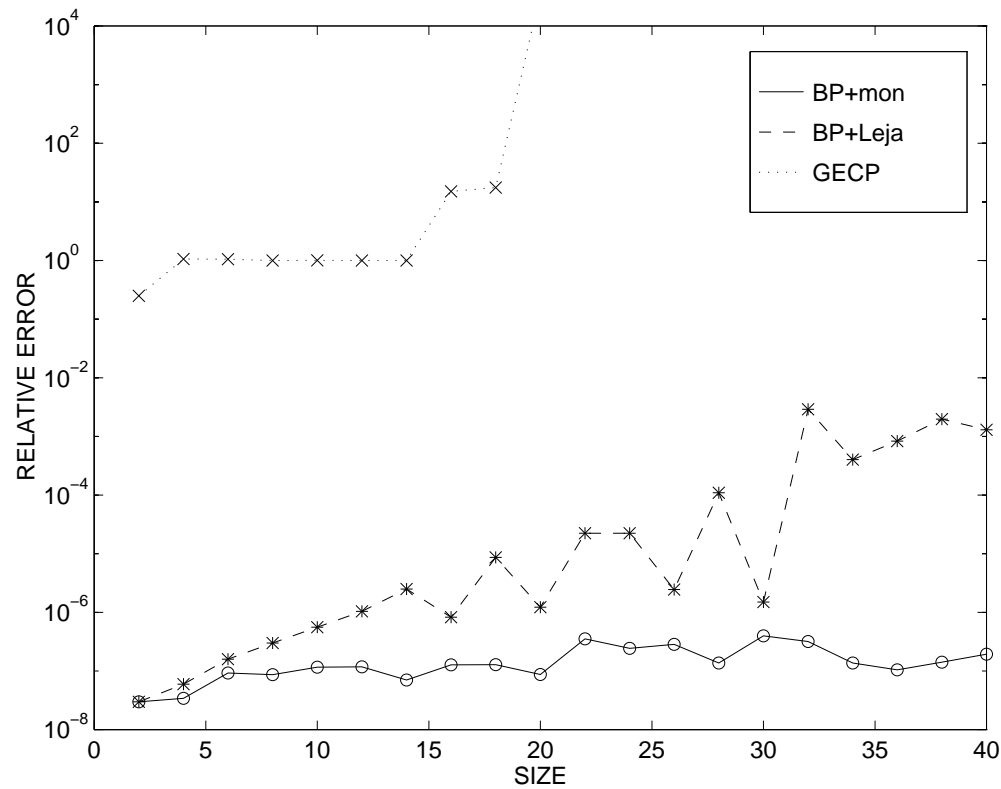
▣▣▣▣ Traub algorithm.

1. Nodes chosen randomly in $(0, 1)$.
2. Forward error measured by

$$e = \frac{\|A^{-1} - \widehat{A^{-1}}\|_2}{\|A^{-1}\|_2}$$

Numerical Experiments

Classical Björck-Pereyra & Traub Algorithms



Error Analysis

Björck-Pereyra Algorithm

- ▣▣▣▣ **Higham** (1990) showed that under some conditions (sign-oscillating right-hand-side and monotonic ordering of the nodes, all of which are positive), there is a **provably excellent forward error bound**:

$$\frac{|a - \hat{a}|}{|a|} \leq 5nu + \mathcal{O}(u^2)$$

- ▣▣▣▣ **Conclusion:** Very good forward error is to be expected under some conditions.

Generalizations of these algorithms

Polynomial–Vandermonde matrices

►► **Definition.** For sets of polynomials and nodes, define a **polynomial–Vandermonde matrix**:

$$\begin{aligned}
 x &= \{x_1, x_2, \dots, x_n\} \\
 R &= \{r_0(x), r_1(x), \dots, r_{n-1}(x)\} \\
 &\quad \Downarrow \\
 V_R(x) &= \begin{bmatrix} r_0(x_1) & r_1(x_1) & r_2(x_1) & \cdots & r_{n-1}(x_1) \\ r_0(x_2) & r_1(x_2) & r_2(x_2) & \cdots & r_{n-1}(x_2) \\ r_0(x_3) & r_1(x_3) & r_2(x_3) & \cdots & r_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_0(x_n) & r_1(x_n) & r_2(x_n) & \cdots & r_{n-1}(x_n) \end{bmatrix}
 \end{aligned}$$

►► **Note.** If the polynomial system R satisfies nice recurrence relations, then the n^2 entries of $V_R(x)$ can be defined by only $\mathcal{O}(n)$ parameters.

Previous work in fast algorithms extending those for Vandermonde matrices

polynomial–Vandermonde matrix	Björck–Pereyra–type linear system solver	Traub–type inversion algorithm
Vandermonde	Björck-Pereyra(1970)	Traub (1966)
block Vandermonde matrices	Tang-Golub (1981)	
Chebyshev-Vandermonde	Reichel-Opfer (1991)	Gohberg-Olshevsky (1994)
three-term Vandermonde	Higham (1988,90)	Calvetti-Reichel (1993)
Szegö-Vandermonde	BEGKO (2006)	Olshevsky (2001)

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three-term Vandermonde	Higham (1988,90)	Calvetti-Reichel (1993)
Szegö-Vandermonde	BEGKO (2006)	Olshevsky (2001)
Quasiseparable-Vandermonde	BEGKO (2007)	????????????????

Quasiseparable Matrices

⇒ **Definition.** A matrix C is $(H, 1)$ -**quasiseparable** if it is upper Hessenberg and

$$\max \text{Rank} C_{12} = 1$$

where the maxima are taken over all symmetric partitions of the form

$$C = \left[\begin{array}{c|c} * & C_{12} \\ \hline * & * \end{array} \right]$$

⇒ **Previous Work.** Gohberg-Kaashoek-Lerer, Dewilde, Gohberg-Eidelman, Van Barel et al, Tyrtyshnikov et al, Bini et al, Gu-Chandrasekaran et al.

Important Special Cases

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

▣▶ The system of polynomials $r_k(x) = \det(xI - C_{k \times k})$ associated with C is the **monomials** with recurrence relations

$$r_k(x) = xr_{k-1}(x)$$

▣▶ The matrix C is $(H, 1)$ -**quasiseparable**.

Important Special Cases

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Important Special Cases

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Important Special Cases

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Important Special Cases

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Important Special Cases

Tridiagonal

$$C = \begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

- ▣ The system of polynomials $r_k(x) = \det(xI - C_{k \times k})$ associated with C are **real orthogonal polynomials** with recurrence relations

$$r_k(x) = \frac{1}{q_k}(x - d_k)r_{k-1}(x) - \frac{g_{k-1}}{q_k}r_{k-2}(x)$$

- ▣ The matrix C is $(H, 1)$ -**quasiseparable**.

Important Special Cases

Tridiagonal

$$C = \begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

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Important Special Cases

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Important Special Cases

Tridiagonal

$$C = \begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

Important Special Cases

Unitary Hessenberg

$$C = \begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 & -\rho_0^* \mu_1 \mu_2 \mu_3 \mu_4 \rho_5 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 & -\rho_1^* \mu_2 \mu_3 \mu_4 \rho_5 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 & -\rho_2^* \mu_3 \mu_4 \rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 & -\rho_3^* \mu_4 \rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^* \rho_5 \end{bmatrix}$$

➡ The system of polynomials $r_k(x) = \det(xI - C_{k \times k})$ associated with C are the **Szegő polynomials** with recurrence relations

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ xr_{k-1}(x) \end{bmatrix}$$

➡ The matrix C is $(H, 1)$ -**quasiseparable**.

Important Special Cases

Unitary Hessenberg

$$C = \begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 & -\rho_0^* \mu_1 \mu_2 \mu_3 \mu_4 \rho_5 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 & -\rho_1^* \mu_2 \mu_3 \mu_4 \rho_5 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 & -\rho_2^* \mu_3 \mu_4 \rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 & -\rho_3^* \mu_4 \rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^* \rho_5 \end{bmatrix}$$

Important Special Cases

Unitary Hessenberg

$$C = \begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 & -\rho_0^* \mu_1 \mu_2 \mu_3 \mu_4 \rho_5 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 & -\rho_1^* \mu_2 \mu_3 \mu_4 \rho_5 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 & -\rho_2^* \mu_3 \mu_4 \rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 & -\rho_3^* \mu_4 \rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^* \rho_5 \end{bmatrix}$$

Important Special Cases

Unitary Hessenberg

$$C = \begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 & -\rho_0^* \mu_1 \mu_2 \mu_3 \mu_4 \rho_5 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 & -\rho_1^* \mu_2 \mu_3 \mu_4 \rho_5 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 & -\rho_2^* \mu_3 \mu_4 \rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 & -\rho_3^* \mu_4 \rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^* \rho_5 \end{bmatrix}$$

Important Special Cases

Unitary Hessenberg

$$C = \begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 & -\rho_0^* \mu_1 \mu_2 \mu_3 \mu_4 \rho_5 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 & -\rho_1^* \mu_2 \mu_3 \mu_4 \rho_5 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 & -\rho_2^* \mu_3 \mu_4 \rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 & -\rho_3^* \mu_4 \rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^* \rho_5 \end{bmatrix}$$

A generator representation

► The following matrix is $(H, 1)$ -**quasiseparable**:

$$\begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ 0 & 0 & p_4 q_3 & d_4 & g_4 h_5 \\ 0 & 0 & 0 & p_5 q_4 & d_5 \end{bmatrix}$$

► This **generator representation** exists for any $(H, 1)$ -**quasiseparable** matrix.

$$(H, 1)\text{-quasiseparable} \iff \text{Generator representation}$$

► Algorithms operating on generators allow use of **low storage**.

A Björck–Pereyra–like algorithm for quasiseparable–Vandermonde matrices

Joint work with Yuli Eidelman, Israel Gohberg, Israel Koltracht, & Vadim Olshevsky

Quasiseparable-Vandermonde matrices

⇒ **Definition.** A **Quasiseparable–Vandermonde matrix** is of the form

$$V_R = \begin{bmatrix} r_0(x_1) & r_1(x_1) & r_2(x_1) & \cdots & r_{n-1}(x_1) \\ r_0(x_2) & r_1(x_2) & r_2(x_2) & \cdots & r_{n-1}(x_2) \\ r_0(x_3) & r_1(x_3) & r_2(x_3) & \cdots & r_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_0(x_n) & r_1(x_n) & r_2(x_n) & \cdots & r_{n-1}(x_n) \end{bmatrix}$$

where the polynomials $r_k(x) = \det(xI - C_{k \times k})$ correspond to an $(H, 1)$ -**quasiseparable** matrix C .

Björck–Pereyra–like algorithm for quasiseparable–Vandermonde matrices

Like the Björck–Pereyra algorithm, the generalization is based on the formula

$$V_R^{-1} = U_1^{-1} \cdots U_{n-1}^{-1} L_{n-1}^{-1} \cdots L_1^{-1},$$

with

$$U_k^{-1} = \text{diag}\left\{ I_{k-1}, \begin{bmatrix} \frac{1}{\alpha_0} & & & \\ 0 & \boxed{C - x_k I} & & \\ \vdots & & \ddots & \\ 0 & & & \frac{1}{\alpha_{n-k}} \end{bmatrix} \right\}$$

$$L_k^{-1} = \begin{bmatrix} I_{k-1} & & & \\ \hline & 1 & & \\ & & \frac{1}{x_{k+1} - x_k} & \\ & & & \ddots \\ & & & & \frac{1}{x_n - x_k} \end{bmatrix} \begin{bmatrix} I_{k-1} & & & \\ \hline & 1 & & \\ & -1 & 1 & \\ & \vdots & & \ddots \\ & -1 & & & 1 \end{bmatrix}$$

Structure of the matrix C involved in U_k^{-1}

- Comparing this matrix U_k^{-1} with that of the classical Björck–Pereyra algorithm

$\begin{bmatrix} \frac{1}{\alpha_0} & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \\ \hline 0 & & \dots & 0 & \frac{1}{\alpha_{n-k}} \end{bmatrix}$	$\begin{bmatrix} 1 & -x_k & & & \\ 0 & 1 & -x_k & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & 1 & -x_k \\ \hline 0 & 0 & \dots & 0 & 1 \end{bmatrix}$
More General U_k^{-1}	Classical U_k^{-1}

- When this algorithm is derived for the **monomials**, the resulting structure of C is the **lower shift matrix**, related to the monomials.
- When this algorithm is derived for $(H, 1)$ –**quasiseparable polynomials**, the resulting structure of C is $(H, 1)$ –**quasiseparable**.
- Fast multiplication by $(H, 1)$ –quasiseparable matrices?**

Complexity of the Björck–Pereyra–like algorithm

Proposition. An $(H, 1)$ –quasiseparable matrix C admits the decomposition

$$C = \mathbf{L} + \mathbf{U}$$

for a **lower–bidiagonal matrix** L and

$$\mathbf{U} = \begin{bmatrix} \mathbf{g}_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbf{g}_{n-1} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & & & \\ \vdots & \tilde{\mathbf{B}}^{-1} & & \\ 0 & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \mathbf{h}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{h}_n \end{bmatrix}$$

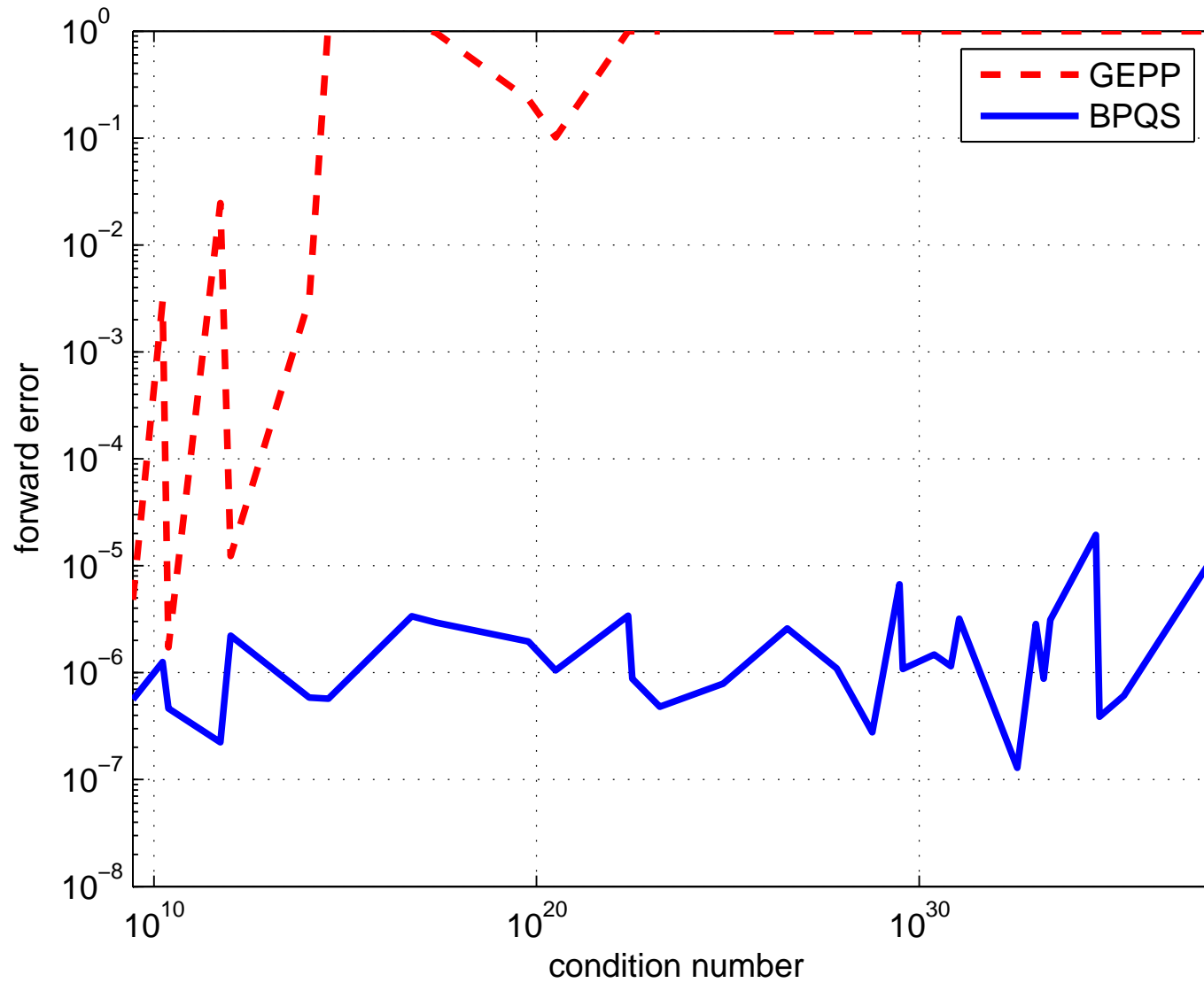
and

$$\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{I} & -\mathbf{b}_2 & & \\ & \ddots & \ddots & \\ & & \mathbf{I} & -\mathbf{b}_{n-1} \\ & & & \mathbf{I} \end{bmatrix}$$

Complexity of the Björck–Pereyra–like algorithm

- ▶▶▶ Thus, in the **quasiseparable** case, C can be multiplied by a vector at the cost of a multiplication by a bidiagonal matrix, two diagonal scalings, and a back–substitution with a bidiagonal matrix.
- ▶▶▶ This leads to an $\mathcal{O}(n)$ algorithm.
- ▶▶▶ This implementation coincides with the algorithm derived differently by **Eidelman and Gohberg** (1999).
- ▶▶▶ Thus the cost of the Björck–Pereyra–like algorithm is $\mathcal{O}(n^2)$ arithmetic operations.

Numerical Illustrations



Classification of $(H, 1)$ -quasiseparable matrices

Joint work with Yuli Eidelman, Israel Gohberg, & Vadim Olshevsky

Efficient recurrence relations for quasiseparable polynomials

Matrices A	Polynomials $r_k(x)$
Lower shift matrix	Monomials
Tridiagonal matrix	Chebyshev polynomials
Tridiagonal matrix	Real-orthogonal polynomials
Unitary Hessenberg matrix	Szegő polynomials
Quasiseparable matrix	Quasiseparable polynomials

$$\mathbf{r}_k(x) = \det(xI - A)_{(\mathbf{k} \times \mathbf{k})}$$

Efficient recurrence relations for quasiseparable polynomials

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Recurrence relations

$$r_k(x) = \mathbf{x} \cdot r_{k-1}(x)$$

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Recurrence relations

$$r_k(x) = 2\mathbf{x} \cdot r_{k-1}(x) - r_{k-2}(x)$$

Efficient recurrence relations for quasiseparable polynomials

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Quasiseparable matrix	Quasiseparable polynomials

Recurrence relations

$$r_k(x) = \frac{1}{q_k} (\mathbf{x} - d_k) r_{k-1}(x) - \frac{g_{k-1}}{q_k} r_{k-2}(x)$$

Efficient recurrence relations for quasiseparable polynomials

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Recurrence relations (2-term)

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ \mathbf{x}r_{k-1}(x) \end{bmatrix}$$

Efficient recurrence relations for quasiseparable polynomials

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Recurrence relations (3-term)

$$r_k(x) = -\frac{1}{\mu_k} \left(\mathbf{x} + \frac{\rho_k}{\rho_{k-1}} \right) r_{k-1}(x) - \left(\frac{\rho_k \mu_{k-1}}{\rho_{k-1} \mu_k} \right) \mathbf{x} \cdot r_{k-2}(x)$$

Efficient recurrence relations for quasiseparable polynomials

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Recurrence relations

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Three-term recurrence relations.

Real-orthogonal polynomials: $\beta_k = 0$

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - \gamma_k r_{k-2}(x)$$

Szegő polynomials (orthogonal on the unit circle): $\gamma_k = 0$

$$r_k(x) = \left(\frac{1}{\mu_k} x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k} \right) r_{k-1}(x) - \left(\frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x \right) r_{k-2}(x)$$

Consider the class of polynomials satisfying more general three-term recurrence relations of the form

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - (\beta_k x + \gamma_k) r_{k-2}(x)$$

Three-term recurrence relations.

Real-orthogonal polynomials: $\beta_k = 0$

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - \gamma_k r_{k-2}(x)$$

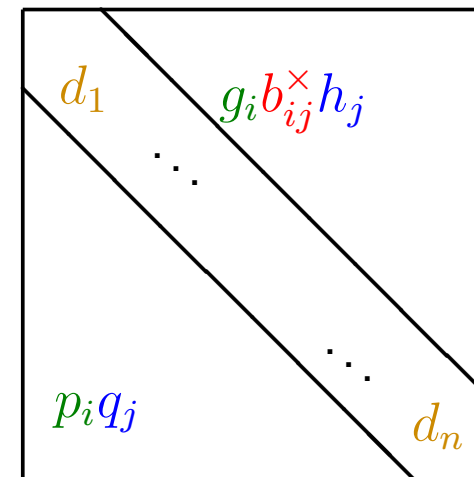
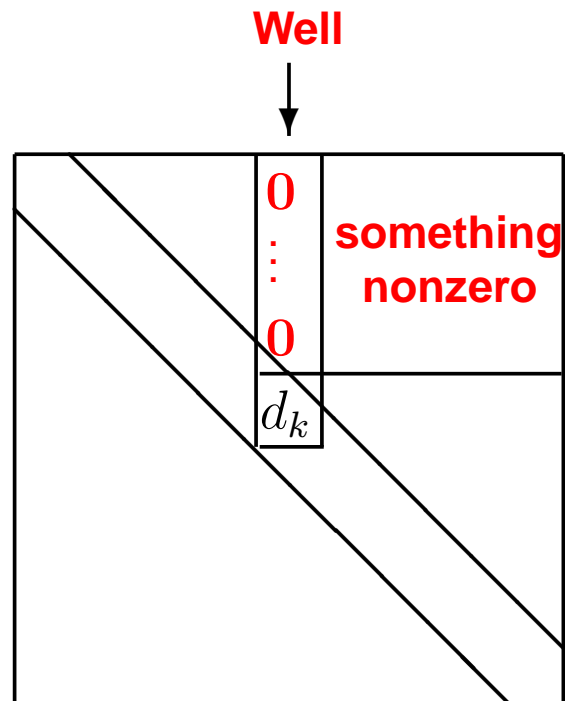
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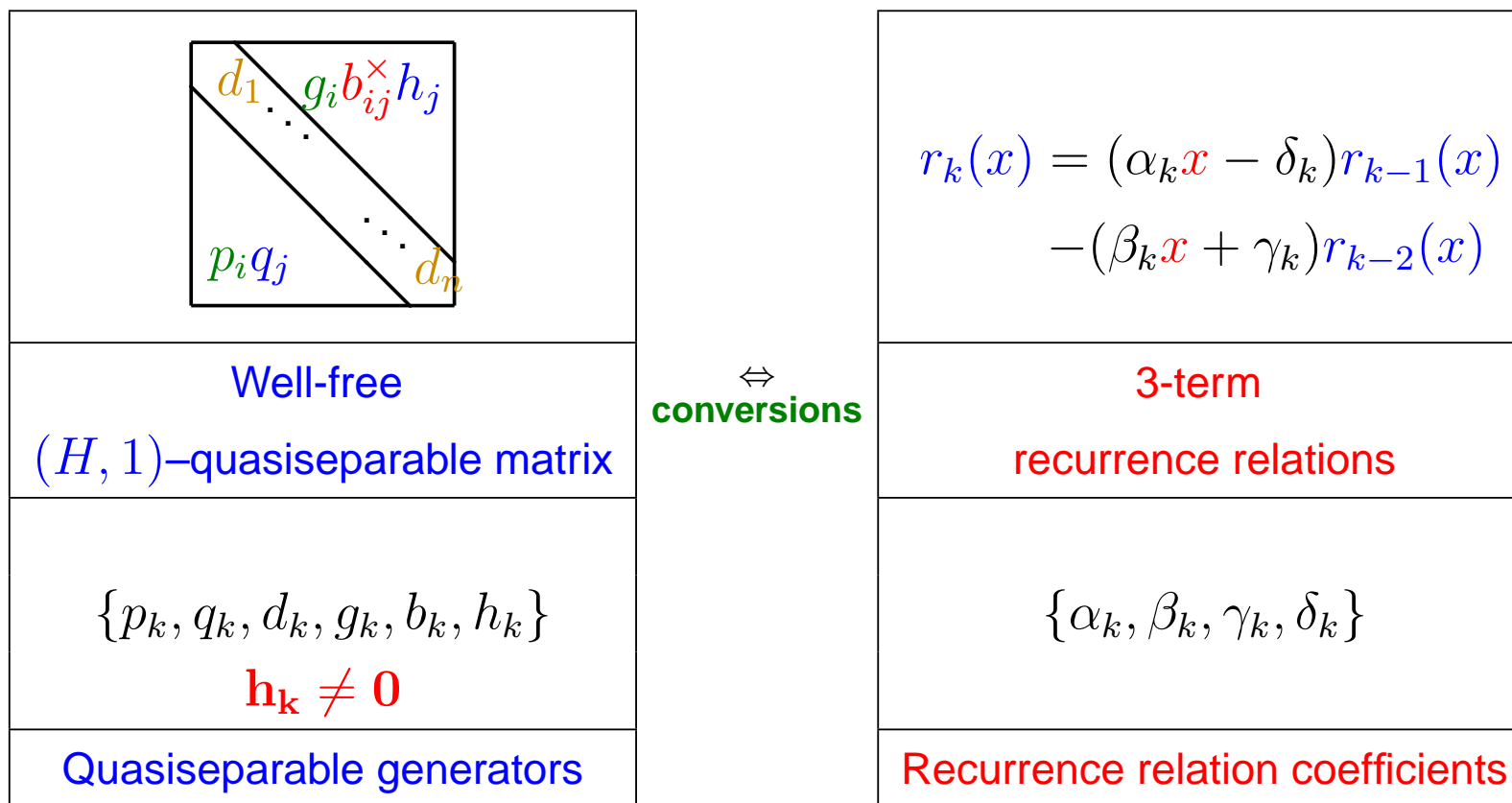
First matrix class. Well-free matrices



$$\text{Well-free} \Leftrightarrow h_j \neq 0$$

- ▣ Irreducible tridiagonal are well-free.
- ▣ Unitary Hessenberg are well-free.

Well-free matrices & 3-term recurrence relations. Equivalence



Restrictions: $h_k \neq 0$

Subclasses of $(H, 1)$ -quasiseparable matrices

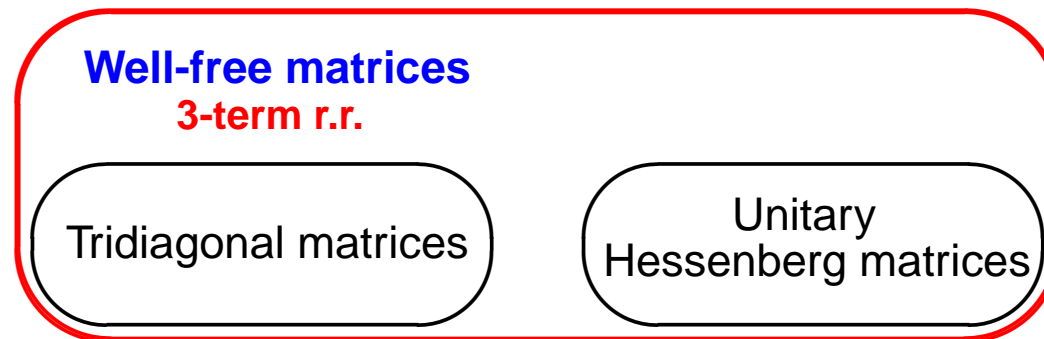
Corresponding recurrence relations

Tridiagonal matrices

Unitary
Hessenberg matrices

Subclasses of $(H, 1)$ -quasiseparable matrices

Corresponding recurrence relations



Second matrix class. Semiseparable matrices

▣▣▣ R is called **order** (r_L, r_U) -**semiseparable** if for some **small** r_L, r_U we have

$$R = D + \text{tril}(R_L) + \text{triu}(R_U),$$

where $\text{rank}R_L = r_L$, $\text{rank}R_U = r_U$, with some R_L, R_U .

▣▣▣ Example of order $(1, 1)$ -**semiseparable**:

$$R_L = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 \end{bmatrix}, R_U = \begin{bmatrix} c_1d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ c_2d_1 & c_2d_2 & c_2d_3 & c_2d_4 \\ c_3d_1 & c_3d_2 & c_3d_3 & c_3d_4 \\ c_4d_1 & c_4d_2 & c_4d_3 & c_4d_4 \end{bmatrix}$$

$$R = \begin{bmatrix} d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ a_2b_1 & d_2 & c_2d_3 & c_2d_4 \\ a_3b_1 & a_3b_2 & d_3 & c_3d_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & d_4 \end{bmatrix}$$

Second matrix class. Semiseparable matrices

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$$R = \begin{bmatrix} d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ a_2b_1 & d_2 & c_2d_3 & c_2d_4 \\ a_3b_1 & a_3b_2 & d_3 & c_3d_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & d_4 \end{bmatrix}$$

Second matrix class. Semiseparable matrices

▣▣▣ R is called **order** (r_L, r_U) -**semiseparable** if for some **small** r_L, r_U we have

$$R = D + \text{tril}(R_L) + \text{triu}(R_U),$$

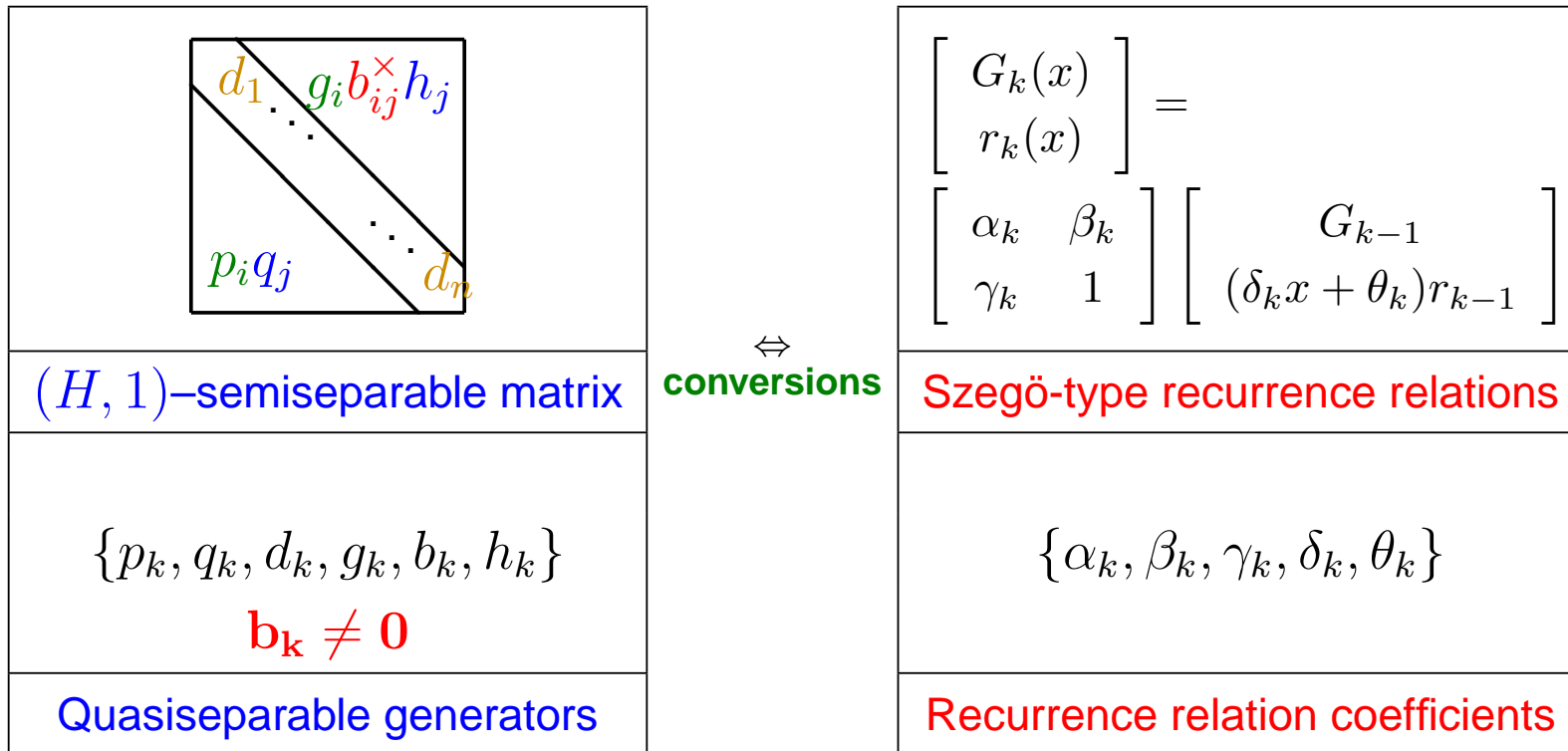
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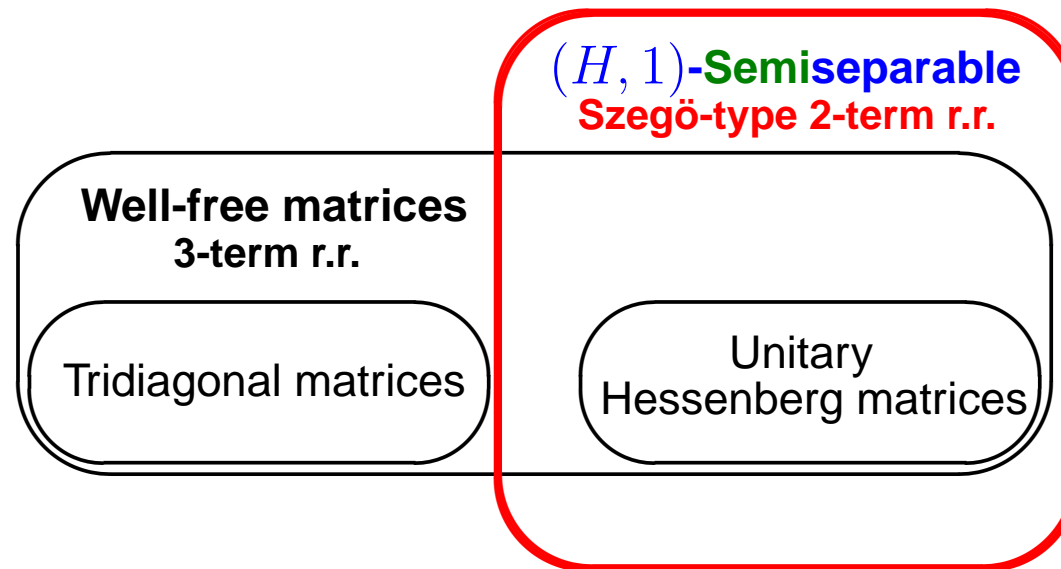
Semiseparable matrices & Szegő-type 2-term recurrence relations. Equivalence



Restrictions: $b_k \neq 0$

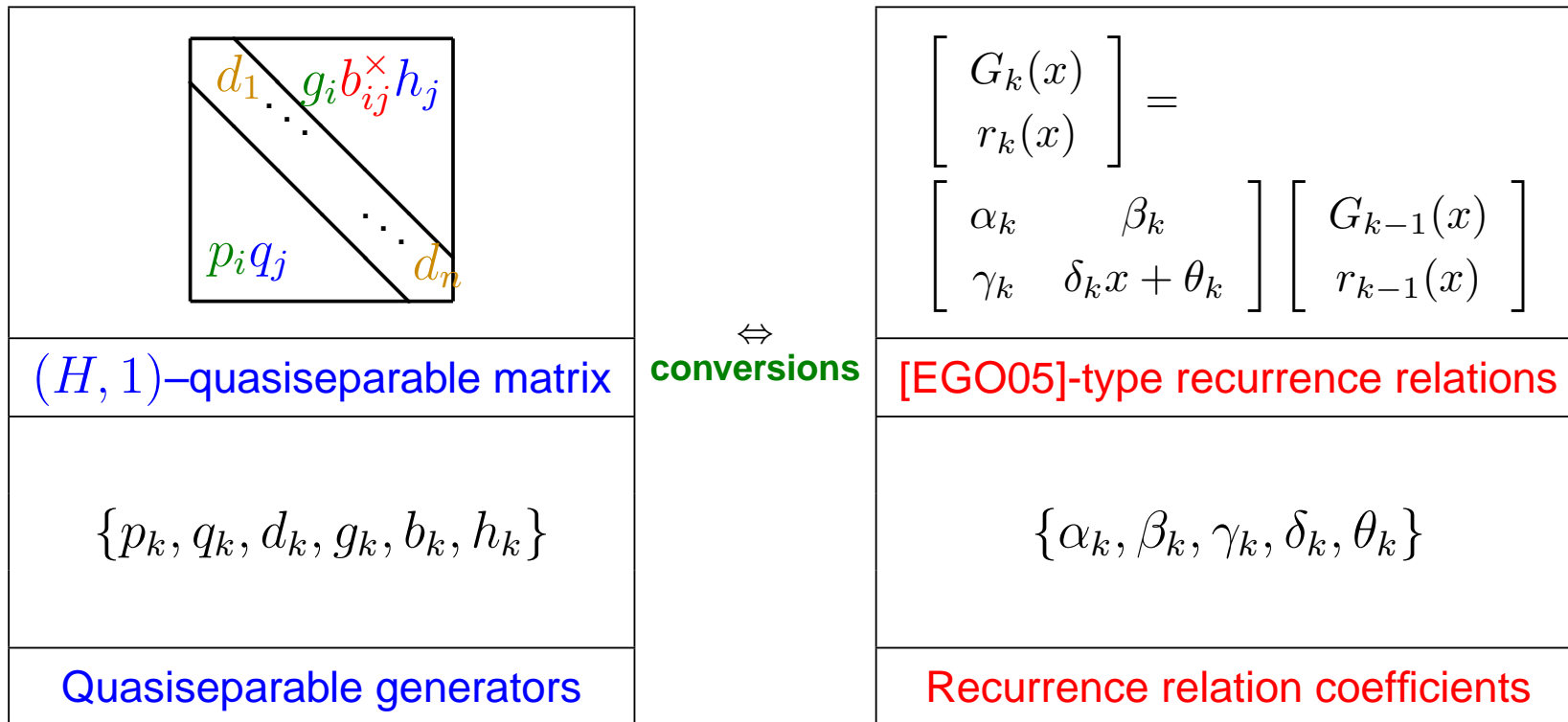
Subclasses of $(H, 1)$ -quasiseparable matrices

Corresponding recurrence relations



Quasiseparable matrices & [EGO05]-type 2-term recurrence relations. Equivalence.

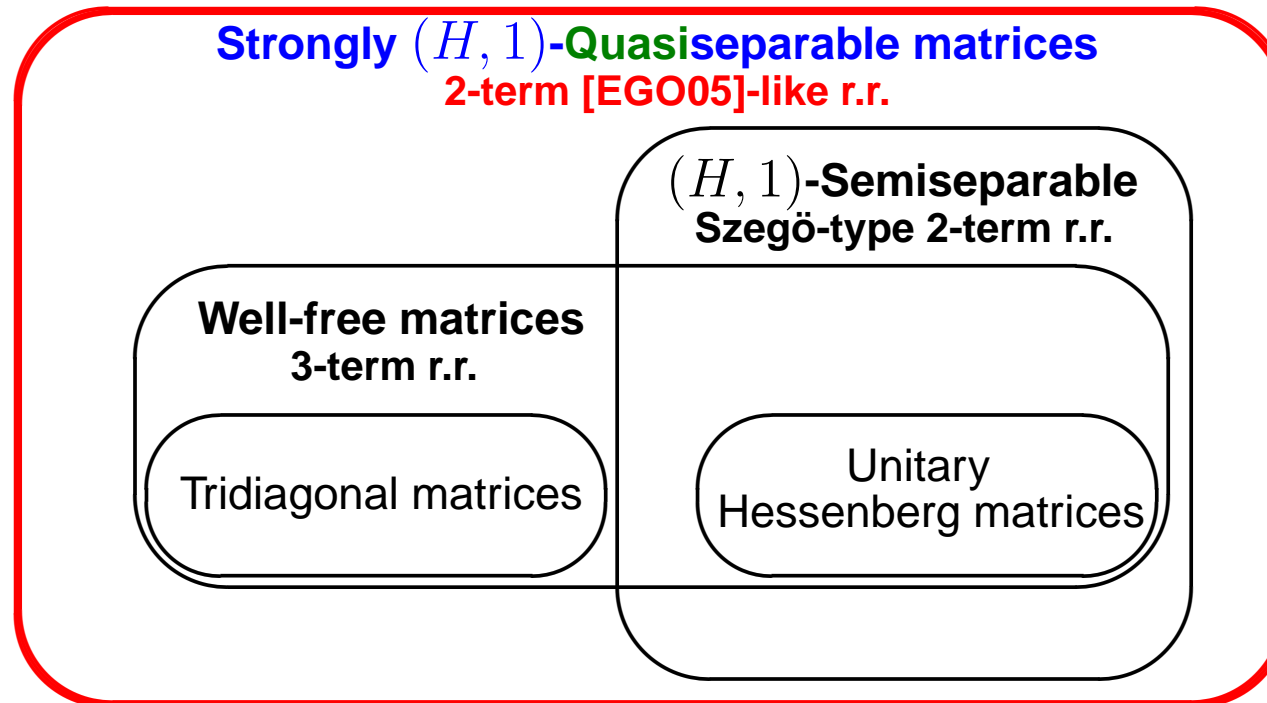
A complete characterization of Hessenberg order-one quasiseparable matrices



Restrictions: NONE.

Full Characterization of $(H, 1)$ -quasiseparable matrices

Corresponding recurrence relations



Efficient recurrence relations for quasiseparable polynomials

Matrices A	Polynomials $r_k(x)$
Lower shift matrix	Monomials
Tridiagonal matrix	Chebyshev polynomials
Tridiagonal matrix	Real-orthogonal polynomials
Unitary Hessenberg matrix	Szegő polynomials
Quasiseparable matrix	Quasiseparable polynomials

Recurrence relations

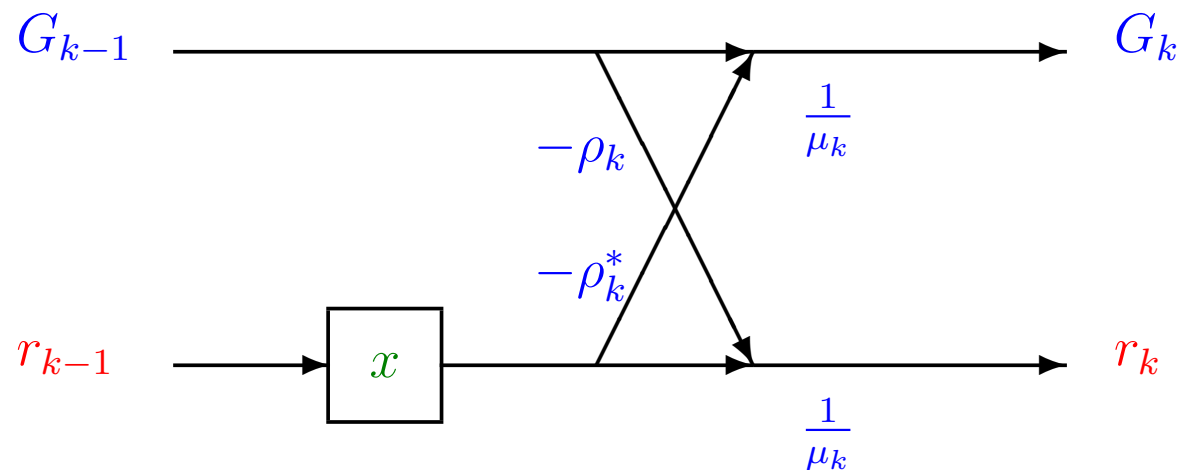
$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k \mathbf{x} + \theta_k)r_{k-1}(x) \end{bmatrix}$$

An Application: Signal flow graphs

Szegő polynomials satisfy well-known two-term recurrence relations,

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ \mathbf{x}r_{k-1}(x) \end{bmatrix}.$$

These can be realized as the well-known **lattice structure**

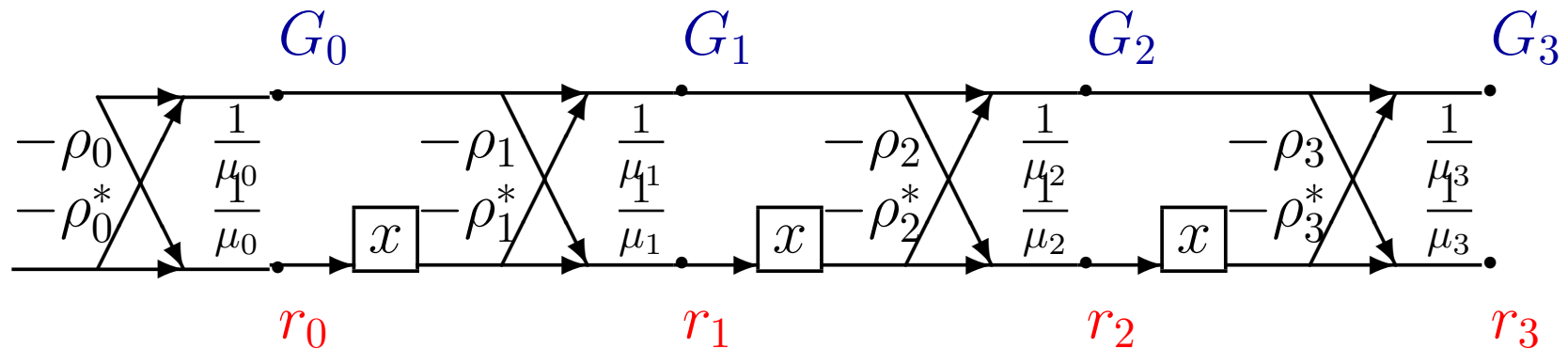


Well-known lattice filter characterization of Unitary Hessenberg

THEOREM: Matrix A is (almost) unitary Hessenberg if and only if polynomials

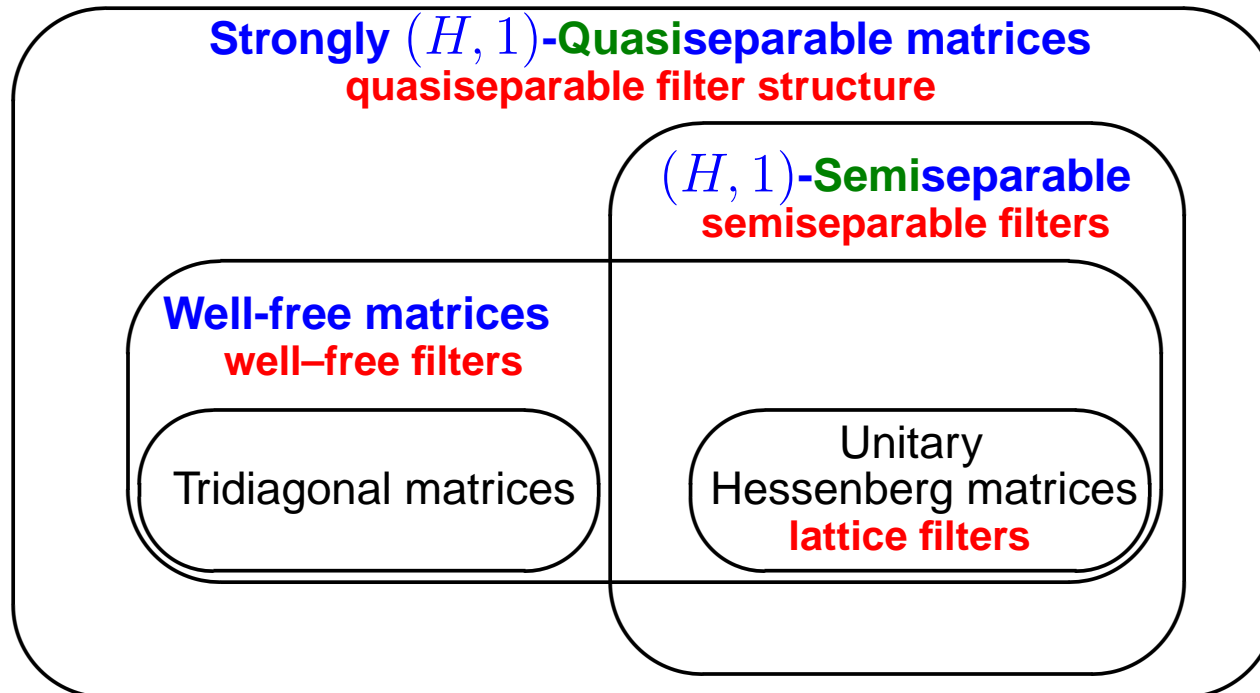
$$r_k(x) = \det(xI - A)_{(k \times k)}$$

admit the following lattice realization (with some conditions).



Subclasses of $(H, 1)$ -quasiseparable matrices

Corresponding digital filter structures

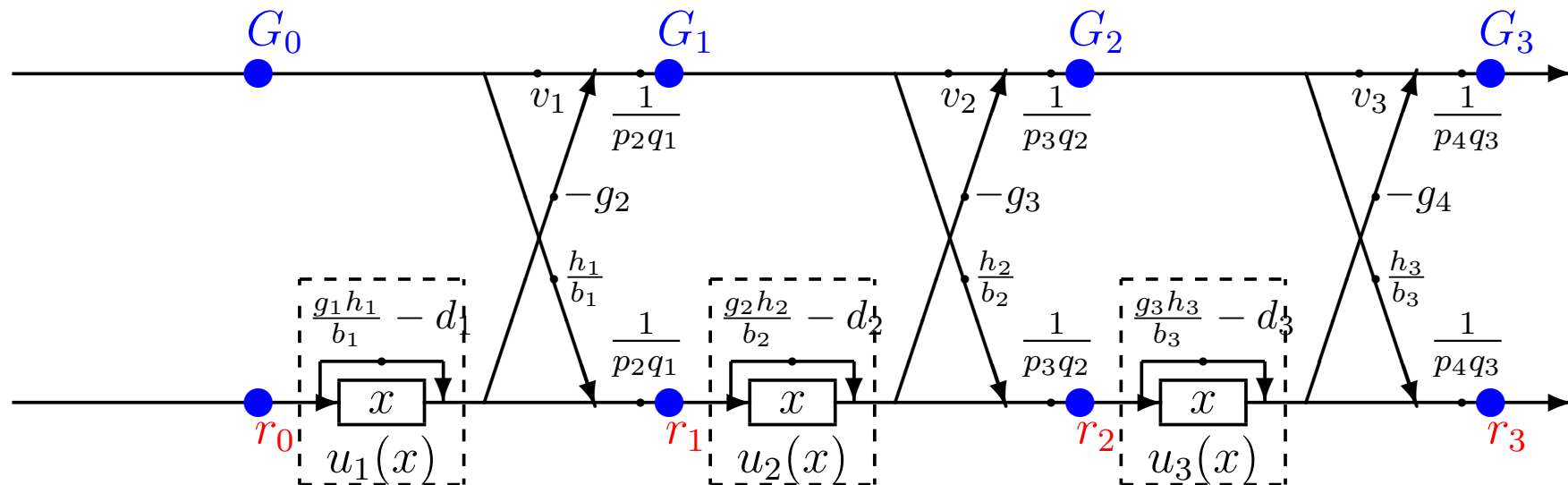


Semiseparable filter structures

THEOREM: Matrix A is $(H, 1)$ -semiseparable if and only if polynomials

$$r_k(x) = \det(xI - A)_{(k \times k)}$$

admit the following lattice-like realization

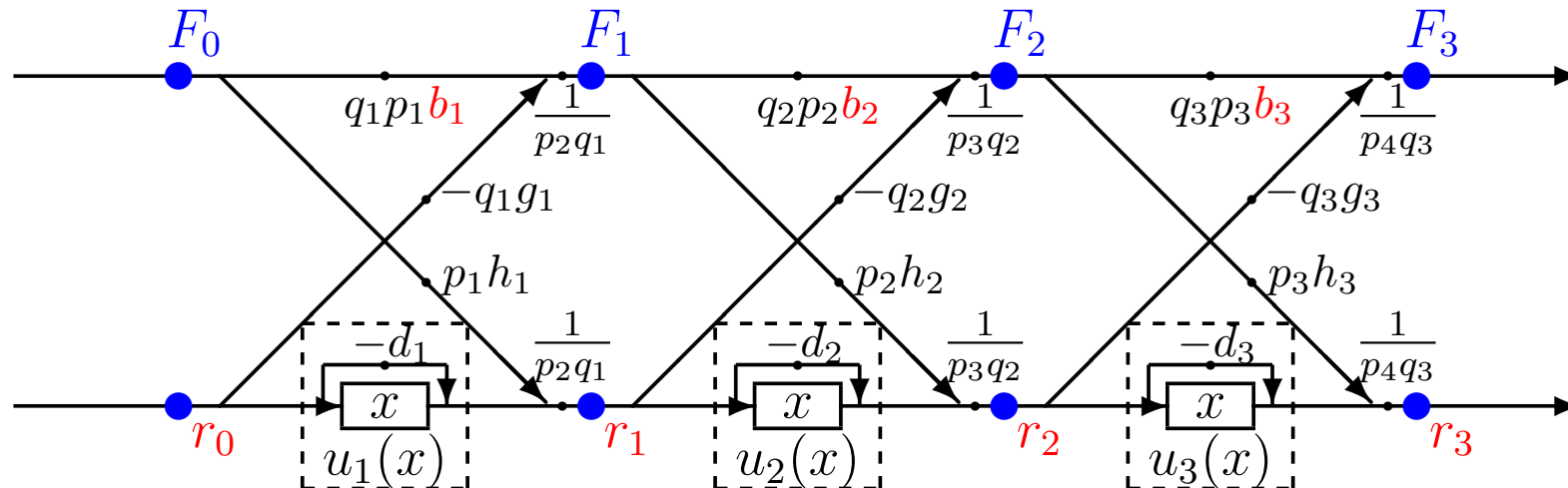


Quasiseparable filter structures

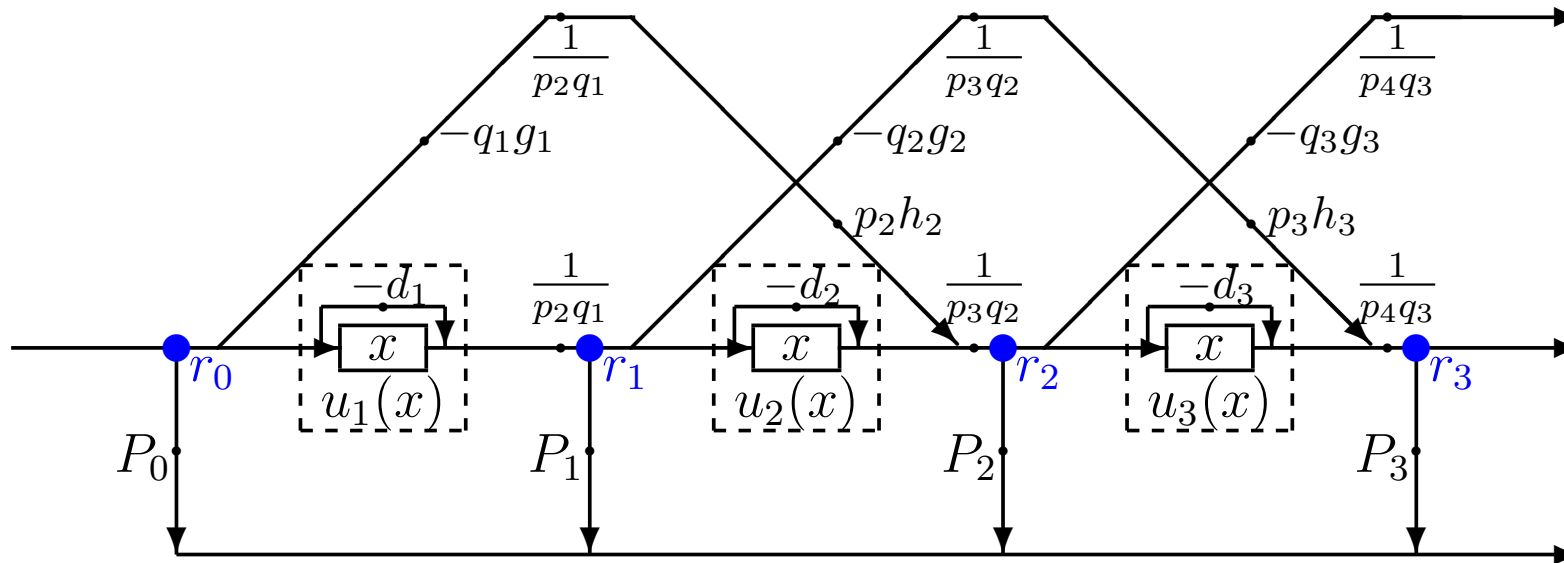
THEOREM: Matrix A is $(H, 1)$ -quasiseparable if and only if polynomials

$$r_k(x) = \det(xI - A)_{(k \times k)}$$

admit the following lattice-like realization



Signal flow graph for real orthogonal polynomials using quasiseparable filter structure



$$\begin{bmatrix}
 d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\
 p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\
 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\
 0 & 0 & p_4 q_3 & d_4 & g_4 h_5 \\
 0 & 0 & 0 & p_5 q_4 & d_5
 \end{bmatrix}, \quad \boxed{\mathbf{b}_k = \mathbf{0}}$$

Quasiseparable matrices and polynomials. Fast and accurate algorithms.

Tom Bella

Department of Mathematics
University of Connecticut
Storrs, CT 06269 USA

Joint work with:

Yuli Eidelman, Israel Gohberg, Israel Koltracht,
Vadim Olshevsky & Eugene Tyrtyshnikov

Hadamard–Sylvester and Pseudo–noise matrices are equivalent

Joint work with Vadim Olshevsky & Lev Sakhnovich

▣▣▣▣ A **Hadamard matrix** is one whose entries are ± 1 and satisfy $H_n^T H_n = nI_n$.

A **Hadamard–Sylvester matrix** is a Hadamard matrix built from the recursion $H_1 = [1]$,

$$H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}.$$

▣▣▣▣ The output of a linear–shift register corresponding to a primitive polynomial is called a **Pseudo–noise sequence**.

A **Pseudo–noise matrix** is a padded, circulant Hankel matrix whose rows are Pseudo–noise sequences.

▣▣▣▣ **Theorem.** Hadamard–Sylvester matrices and Pseudo–noise matrices are **equivalent**; i.e. one can be obtained from the other via row and column permutations.

Structure-preserving perturbations of matrices self-adjoint with respect to an indefinite inner products

Joint work with Vadim Olshevsky & Upendra Prasad

- For a Hermitian, invertible (not necessarily positive definite) matrix H , one defines the **indefinite inner product** via

$$[x, y]_H = (Hx, y) = y^* Hx$$

where (\cdot, \cdot) denotes the Euclidean inner product.

- One defines **self-adjoint with respect to an indefinite inner product** in an analogous way to the definition for classical self-adjoint.

Euclidean Inner Product	Indefinite Inner Product
$(Ax, y) = (x, Ay)$	$[Ax, y]_H = [x, Ay]_H$
$A = A^*$	$HA = A^* H$

Structure-preserving perturbations of matrices self-adjoint with respect to an indefinite inner products

Joint work with Vadim Olshevsky & Upendra Prasad

- ▣ Pairs of matrices (A, H) have a **canonical form** (J, P) [Gohberg–Lancaster–Rodman 1983], with

$$J = J(\lambda_1) \oplus \cdots \oplus J(\lambda_\alpha) \oplus \tilde{J}(\lambda_{\alpha+1}) \oplus \cdots \oplus \tilde{J}(\lambda_\beta)$$

and

$$P = \epsilon_1 P_1 \oplus \cdots \oplus \epsilon_\alpha P_\alpha \oplus P_{\alpha+1} \oplus \cdots \oplus P(\lambda_\beta)$$

where $\lambda_1, \dots, \lambda_\alpha \in \mathbb{R}$, $J(\lambda)$ is a Jordan block, and $\tilde{J}(\lambda) = J(\lambda) \oplus J(\bar{\lambda})$.

- ▣ The matrix that reduces to this canonical form, T such that

$$T^{-1}AT = J, \quad T^*HT = P$$

has columns that are not only a Jordan basis of A , they also bring H into P , and we call such a basis a **canonical Jordan basis of** (A, H) .

Structure-preserving perturbations of matrices self-adjoint with respect to an indefinite inner products

Joint work with Vadim Olshevsky & Upendra Prasad

► **Theorem.** Let $A_0 \in \mathbb{C}^{n \times n}$ be a fixed H_0 -selfadjoint matrix. Let

$$\left\{ \left\{ f_r^{(k,s)} \right\}_{r=0}^{m_k(A_0, \lambda_s) - 1} \right\}_{s=1, k=1}^{s=\beta, k=\dim \ker(A_0 - \lambda_s I)}$$

be a fixed canonical Jordan basis of A_0 . There exist constants $K, \delta > 0$ (depending on A_0 and H_0 only) such that the following assertion holds. For any H -selfadjoint matrix A such that A has the same Jordan structure as A_0 and

$$\|A - A_0\| + \|H - H_0\| < \delta,$$

there exists a canonical Jordan basis

$$\left\{ \left\{ g_r^{(k,s)} \right\}_{r=0}^{m_k(A, \lambda_s) - 1} \right\}_{s=1, k=1}^{s=\beta, k=\dim \ker(A - \lambda_s I)}$$

of A such that

$$\|g_r^{(k,s)} - f_r^{(k,s)}\| \leq K (\|A - A_0\| + \|H - H_0\|)$$

for all k, s, r within their ranges.

Quasiseparable matrices and polynomials. Fast and accurate algorithms.

Tom Bella

Department of Mathematics

University of Connecticut

Storrs, CT 06269 USA

Joint work with:

Yuli Eidelman, Israel Gohberg, Israel Koltracht,

Vadim Olshevsky & Eugene Tyrtyshnikov

Supplemental Slides

What are the coefficients P_k ?

- ▶ The difference between the recurrence relations for the original polynomials and those for the Horner-like polynomials is the presence of the coefficients P_k .
- ▶ P_k is the coefficient of $r_k(x)$ in the decomposition of the **master polynomial** $P(x) = \prod_{i=1}^n (x - x_i)$ into the $\{r_k\}$ basis:

$$\prod_{i=1}^n (x - x_i) = P_0 r_0(x) + P_1 r_1(x) + \cdots + P_n r_n(x)$$

- ▶ These coefficients can be computed in $\mathcal{O}(n^2)$ arithmetic operations.

Special cases of these recurrence relations

Matrix	Polynomial system
Shift matrix	monomials
Tridiagonal matrix	real orthogonal polynomials
Unitary Hessenberg matrix	Szegő polynomials

Special cases of these recurrence relations

Matrix	Polynomial system
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$$\begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ 0 & 0 & p_4 q_3 & d_4 & g_4 h_5 \\ 0 & 0 & 0 & p_5 q_4 & d_5 \end{bmatrix}$$

$$\Downarrow$$

$$p_k = 1, \quad q_k = 1, \quad d_k = 0, \quad g_k = 0, \quad b_k = 0, \quad h_k = 1$$

$$\Downarrow$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Special cases of these recurrence relations

General three-term recurrence relations in the monomial case

$$r_k(x) = (\alpha_k x - \delta_k) \cdot r_{k-1}(x) - (\beta_k x + \gamma_k) \cdot r_{k-2}(x)$$

⇓

$$p_k = 1, \quad q_k = 1, \quad d_k = 0, \quad g_k = 0, \quad b_k = 0, \quad h_k = 1$$

⇓

$$r_k(x) = x \cdot r_{k-1}(x)$$

Special cases of these recurrence relations

Matrix	Polynomial system
Shift matrix	monomials
Tridiagonal matrix	real orthogonal polynomials
Unitary Hessenberg matrix	Szegő polynomials

$$\begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ 0 & 0 & p_4 q_3 & d_4 & g_4 h_5 \\ 0 & 0 & 0 & p_5 q_4 & d_5 \end{bmatrix}$$

 \Downarrow

$$p_k = 1, \quad b_k = 0, \quad h_k = 1$$

 \Downarrow

$$\begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

Special cases of these recurrence relations

General three-term recurrence relations in the real orthogonal polynomial case

$$r_k(x) = (\alpha_k x - \delta_k) \cdot r_{k-1}(x) - (\beta_k x + \gamma_k) \cdot r_{k-2}(x)$$

⇓

$$p_k = 1, \quad b_k = 0, \quad h_k = 1$$

⇓

$$r_k(x) = \frac{1}{q_k} (x - d_k) r_{k-1}(x) - \frac{g_{k-1}}{q_k} r_{k-2}(x)$$

Special cases of these recurrence relations

Matrix	Polynomial system
Shift matrix	monomials
Tridiagonal matrix	real orthogonal polynomials
Unitary Hessenberg matrix	Szegő polynomials

$$\begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ 0 & 0 & p_4 q_3 & d_4 & g_4 h_5 \\ 0 & 0 & 0 & p_5 q_4 & d_5 \end{bmatrix}$$

$$\Downarrow$$

$$p_k = 1, \quad q_k = \mu_k, \quad d_k = -\rho_{k-1}^* \rho_k, \quad g_k = \rho_{k-1}^*, \quad b_k = \mu_{k-1}, \quad h_k = -\mu_{k-1} \rho_k$$

$$\Downarrow$$

$$\begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 & -\rho_0^* \mu_1 \mu_2 \mu_3 \mu_4 \rho_5 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 & -\rho_1^* \mu_2 \mu_3 \mu_4 \rho_5 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 & -\rho_2^* \mu_3 \mu_4 \rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 & -\rho_3^* \mu_4 \rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^* \rho_5 \end{bmatrix}$$

Special cases of these recurrence relations

Szegő–type recurrence relations in the Szegő polynomial case

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k x + \theta_k)r_{k-1}(x) \end{bmatrix}$$

⇓

$$p_k = 1, \quad q_k = \mu_k, \quad d_k = -\rho_{k-1}^* \rho_k, \quad g_k = \rho_{k-1}^*, \quad b_k = \mu_{k-1}, \quad h_k = -\mu_{k-1} \rho_k$$

⇓

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ x r_{k-1}(x) \end{bmatrix}$$