The Second Undergraduate Level Course in Linear Algebra

Steven Leon

University of Massachusetts Dartmouth

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Outline of Talk

- Linear Algebra Curriculum Study Group (LACSG) Recommendations
- LACSG recommended first course topics
- Core topics for a second course
- Numerical Linear Algebra
- Undergraduate explorations and research in linear algebra

Linear Algebra Curriculum Study Group

- Organizers:
  David Carlson, Charles R. Johnson, David C. Lay, and Duane Porter
- The William and Mary Workshop Panel:
  Glenn Adamson, Paul Bengtson, James Bunch, David Carlson, Jane Day, Guershon Harel, Roger Horn

LACSG Recommendations

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4. Faculty should use technology.
5. At least one follow-up course should be required.

The LACSG Core Course – Topics

Core topics

1. Matrix Addition and Multiplication
2. Systems of Linear Equations
3. Determinants
4. Properties of $\mathbb{R}^n$
5. Eigenvalues and Eigenvectors
6. More on Orthogonality
7. Supplementary Topics

1. Matrix Addition and Multiplication

- matrix addition
- scalar multiplication
- matrix multiplication
- transposition
- associativity of matrix multiplication
- operations with partitioned matrices

2. Systems of Linear Equations

- Gaussian elimination
- elementary matrices
- echelon and reduced row echelon form
- existence and uniqueness of solutions
- matrix inverses
- row reduction interpreted as LU-factorization

3. Determinants

- cofactor expansion
- determinants and row operations
- $\det AB = \det A \det B$
- Cramer's Rule

4. Properties of $\mathbb{R}^n$

- linear combinations: linear dependence and independence
- bases of $\mathbb{R}^n$
- subspaces of $\mathbb{R}^n$: spanning set, basis, dimension, row space and column space, null space
- matrices of linear transforms
- rank: row rank=column rank, products, connections with invertible submatrices
- systems of equations revisited: solution theory, rank+nullity=number of columns
- inner product: length and orthogonality, orthogonal /orthonormal sets and bases, orthogonal matrices

5. Eigenvalues and Eigenvectors

- the equation $Ax = \lambda x$
- characteristic polynomial and identification of some of its coefficients (trace and determinant), algebraic multiplicity of eigenvalues
- eigenspaces, geometric multiplicity
- similarity: distinct eigenvalues and diagonalization
- symmetric matrices: orthogonal diagonalization, quadratic forms

6. More on Orthogonality

- orthogonal projection onto a subspace: Gram-Schmidt orthogonalization and interpretation as a QR factorization
- the least square solutions of inconsistent linear systems, with applications to data-fitting

7. Supplementary Topics

- computational experience
- abstract vector spaces
- linear transforms
- positive definite matrices
- reduction of a symmetric matrix by congruence
- singular value decomposition
- matrix norms

LACSG Recommendations Revisited

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2. The first course should be matrix oriented.
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4. Faculty should use technology.
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There has been considerable progress with regards to the first four LACSG recommendations. Evidence of this comes from comparing the textbooks used today with those of 20 years ago.

Challenges for the Future

- Challenge to cover the entire LACSG core curriculum in a single one semester course. At some universities this challenge has been increasing in recent years.
- The Supplementary Topics (Unit 7) are generally deferred to a second course.
- At many universities second courses are offered but not required.

Core topics for the second course

- Abstract vector spaces: function spaces and inner products
- Special classes of matrices: orthogonal and unitary, Hermitian, normal, positive definite.
- Singular value decomposition and other matrix factorizations
- Matrix norms
- The matrix exponential

A Numerical Linear Algebra Course

- Matrix norms and condition numbers
- Numerical algorithms for linear systems, least squares problems, and eigenvalue problems
- Numerical rank and the singular value decomposition
- Householder transformations
- Modified Gram-Schmidt process
- Sparse matrices and Krylov subspaces

Projects oriented courses

- Students work in teams on exploration/research projects
- Student develop MATLAB code for their projects
- Examples of projects
  - Computer animation
  - Applications to coordinate metrology
  - Digital imaging applications

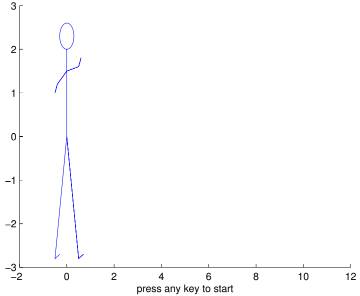
Computer animation

- Contractions, dilations, rotations, and translations are carried as matrix multiplications.
- Homogeneous coordinate system

$$x = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \quad Ax = \begin{bmatrix} x_1 + a \\ x_2 + b \\ 1 \end{bmatrix}$$

Computer animation

- Project make stick figure walk across screen



- Design video game – Meteor

Meteor video game

>> help meteor

Key(s)	Action
Left arrow or 'j'	Rotate ship counterclockwise 15 degrees
Up arrow or 'k'	Thrust forward
Right arrow or 'l'	Rotate ship clockwise 15 degrees
Down arrow or ';'	Apply air brakes
Spacebar	Fire shot
'f'	Flip ship (Turn ship 180 degrees)
'h'	Send ship to the home (start) position
'p'	Pause the game
Escape	Quit the game

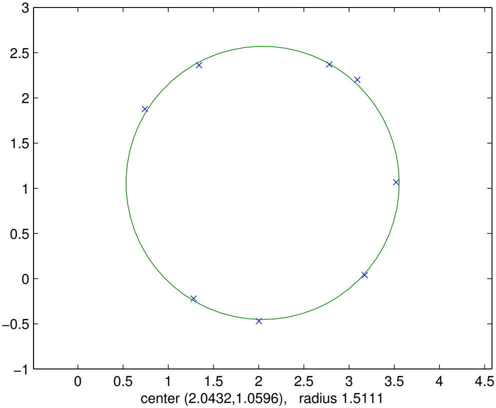
Meteor game description

Each meteoroid is worth 100 points. The speed and direction of each meteoroid are randomly chosen. As levels are completed, the nominal meteoroid speed increases proportionally. After level 2, the deploy time (how long before a new meteoroid is visible) and the number of meteoroids per level are held constant.

Coordinate Metrology Problems

- Write MATLAB program to fit a circle to data points
- Find the line that gives the best geometric (or orthogonal) fit to data points
- Find the best rectangular fit to a set of points.
- Other types of geometric fits

Output from Circlefit MATLAB function



Finding the line that gives best geometric fit to m data points

Equation of line: $c_1 + c_2 x + c_3 y = 0$ with $c_2^2 + c_3^2 = 1$

Let $\quad r_i = c_1 + c_2 x_i + c_3 y_i \quad i = 1 \ldots m$

Find c_1, c_2, c_3 to minimize $\|\mathbf{r}\|^2$ subject to $c_2^2 + c_3^2 = 1$.

If $A = [\mathbf{e}\ \mathbf{x}\ \mathbf{y}]$ then $\mathbf{r} = A\mathbf{c}$

Solution using QR and SVD factorizations

If $A = QR$ then

$$\|\mathbf{r}\|^2 = \|A\mathbf{c}\|^2 = \|QR\mathbf{c}\|^2 = \|R\mathbf{c}\|^2 = \|R_1\mathbf{c}\|^2$$

where

$$R_1\mathbf{c} = \begin{bmatrix} r_{11}c_1 + r_{12}c_2 + r_{13}c_3 \\ r_{22}c_2 + r_{23}c_3 \\ r_{33}c_3 \end{bmatrix}$$

To minimize $\|R_1\mathbf{c}\|^2$ subject to $c_2^2 + c_3^2 = 1$, let $U\Sigma V^T$ be the singular value decomposition of

$$R_2 = \begin{bmatrix} r_{22} & r_{23} \\ & r_{33} \end{bmatrix},$$

set $(c_2, c_3) = \mathbf{v}_2^T$ and then solve for c_1.

Fitting Parallel and Perpendicular lines

Given two data sets

$$(x_i, y_i),\ i = 1, \ldots, m \quad \text{and} \quad (x_i, y_i),\ i = m+1, \ldots, m+k$$

- Parallel lines: Set

$$\begin{aligned} r_i &= c_1 + 0c_2 + c_3 x_i + c_4 y_i \quad i = 1, \ldots, m \\ r_i &= 0c_1 + c_2 + c_3 x_i + c_4 y_i \quad i = m+1, \ldots, m+k \end{aligned}$$

- Perpendicular lines: Set

$$\begin{aligned} r_i &= c_1 + 0c_2 + c_3 x_i + c_4 y_i \quad i = 1, \ldots, m \\ r_i &= 0c_1 + c_2 + c_3 y_i - c_4 x_i \quad i = m+1, \ldots, m+k \end{aligned}$$

Solve for c_1, c_2, c_3, c_4 using QR and SVD factorizations.

Fitting rectangles

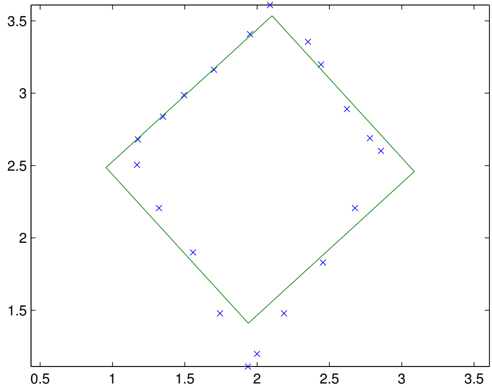
Given 4 data sets each with m points. Set

$$
\begin{aligned}
r_i &= c_1 + 0c_2 + 0c_3 + 0c_4 + c_5 x_i + c_6 y_i \quad && i = 1, \ldots, m \\
r_i &= 0c_1 + c_2 + 0c_3 + 0c_4 + c_5 y_i - c_6 x_i \quad && i = m+1, \ldots, 2m \\
r_i &= 0c_1 + 0c_2 + c_3 + 0c_4 + c_5 x_i + c_6 y_i \quad && i = 2m+1, \ldots, 3m \\
r_i &= 0c_1 + 0c_2 + 0c_3 + c_4 + c_5 y_i - c_6 x_i \quad && i = 3m+1, \ldots, 4m
\end{aligned}
$$

Solve for $c_1, c_2, c_3, c_4, c_5, c_6$ using QR and SVD factorizations.

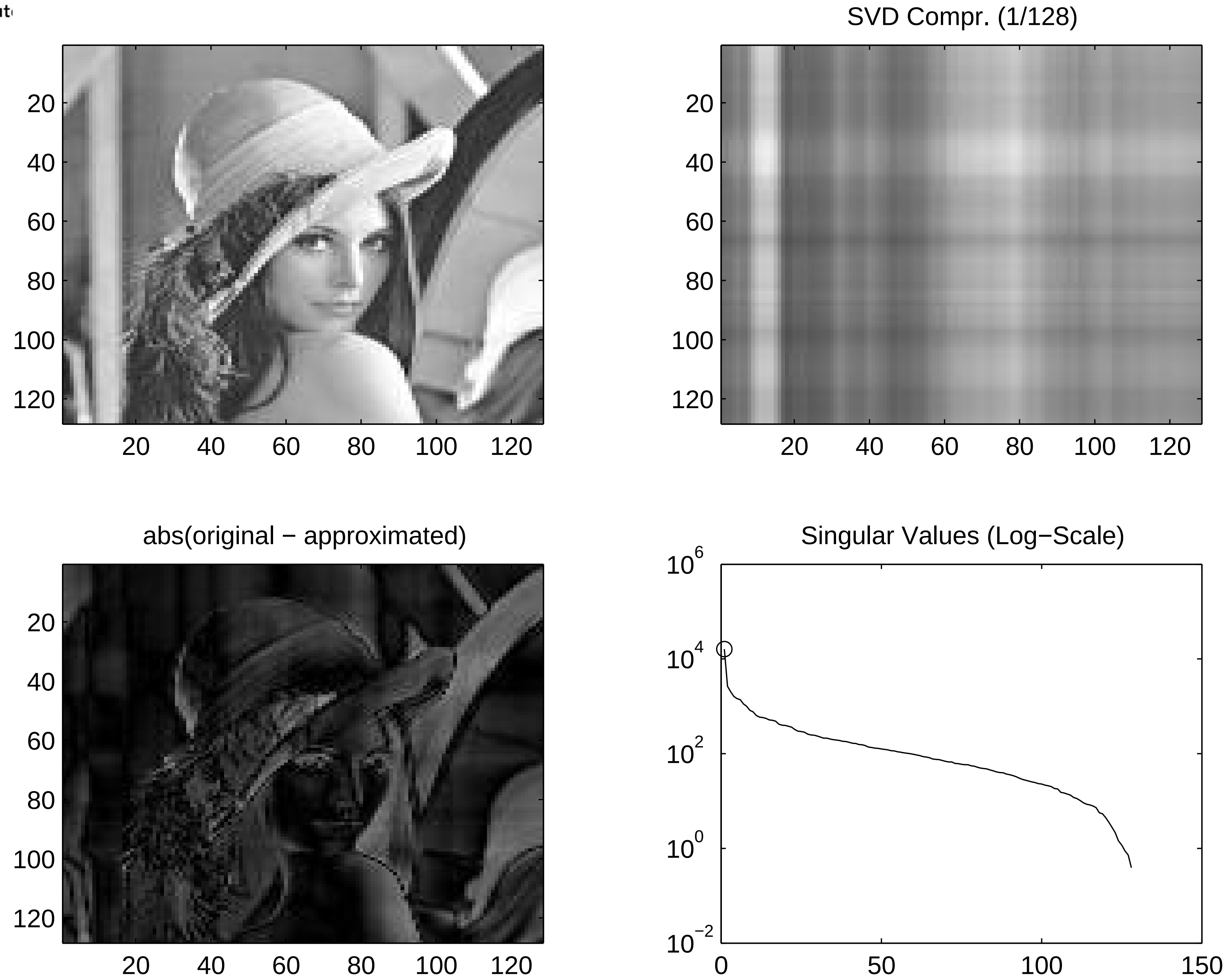
Determine the 4 points where the perpendicular lines intersect.

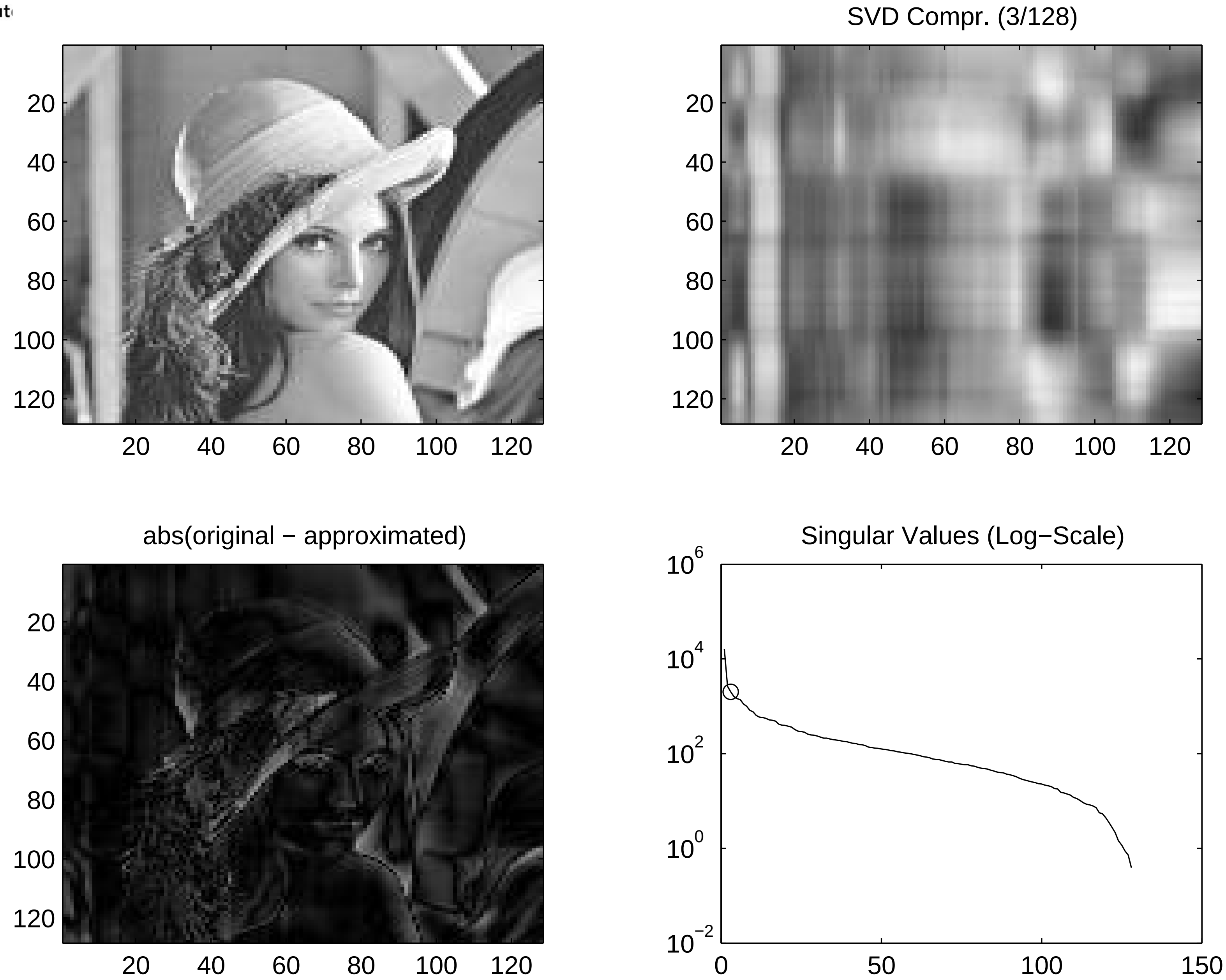
Plot of Rectangle Fit



SVD Image compression

- A black and white picture can be represented as a single $m \times n$ matrix A with integer entries representing the grey levels of the pixels of the image.
- SVD compression of image (low rank approximation of A)
- Singular value decomposition: $A = U\Sigma V^T$,
  $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ and $U^T U = I_m$, $V^T V = I_n$
  Singular values: $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n \geq 0$
- Best rank k approximation to A is $\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_k \mathbf{u}_k \mathbf{v}_k^T$


SVD Compr. (1/128)
abs(original − approximated)
Singular Values (Log−Scale)
20
40
60
80
100
120
0
50
100
150
10^{6}
10^{4}
10^{2}
10^{0}
10^{-2}


SVD Compr. (3/128)
abs(original − approximated)
Singular Values (Log−Scale)
20
40
60
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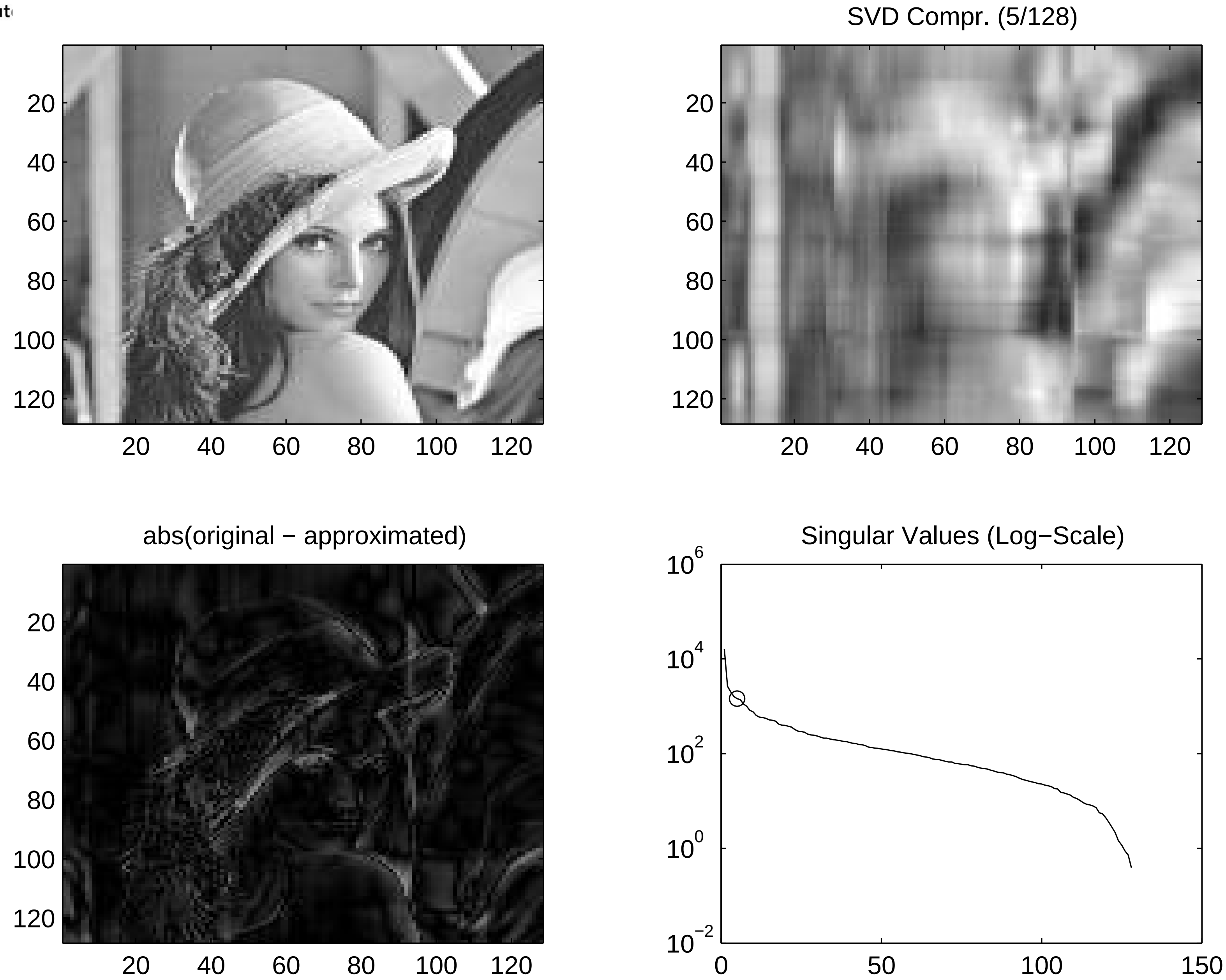
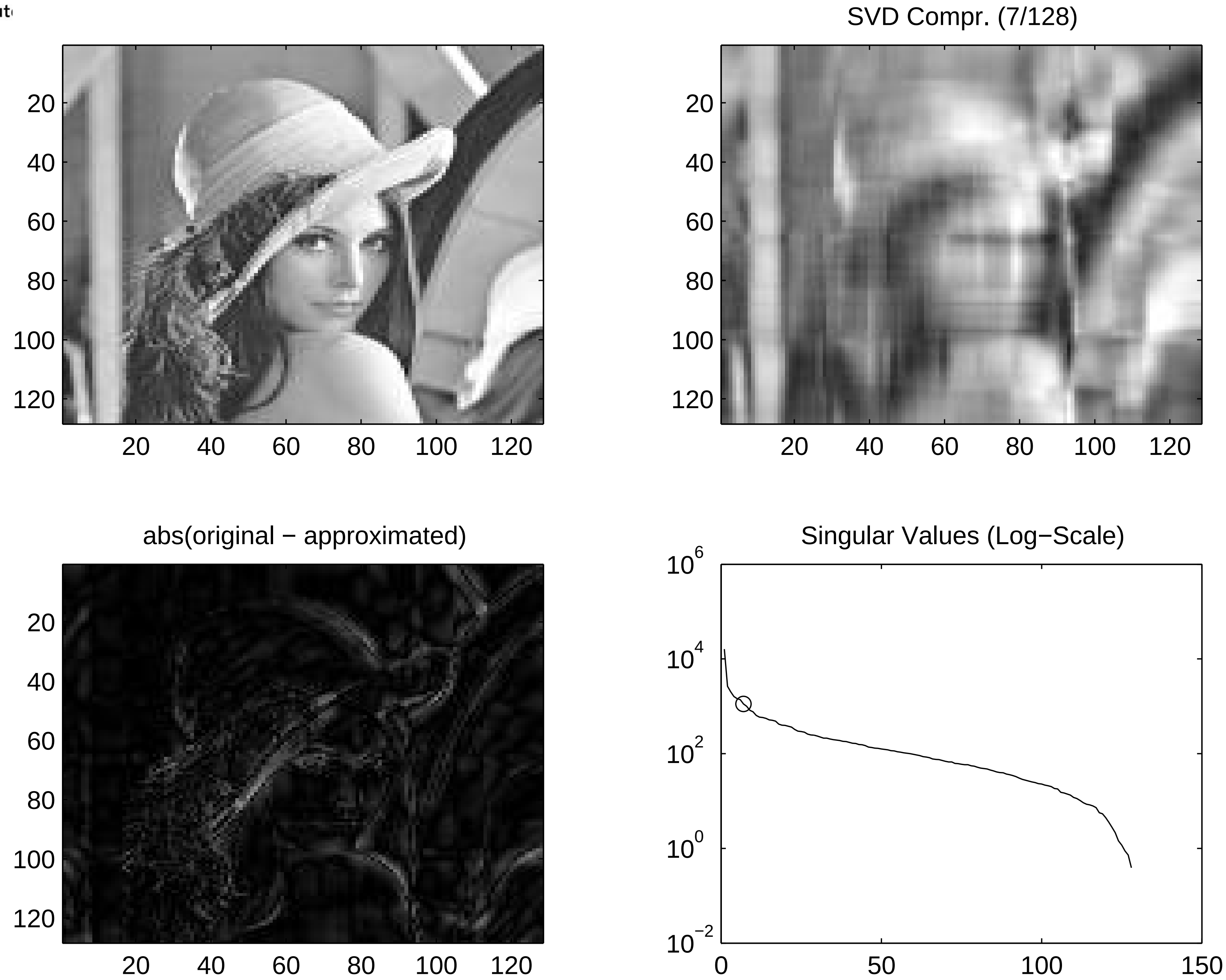
100
120
10^6
10^4
10^2
10^0
10^{-2}
0
50
100
150


SVD Compr. (5/128)
abs(original − approximated)
Singular Values (Log−Scale)
20 40 60 80 100 120
10^6
10^4
10^2
10^0
10^{-2}
0 50 100 150


SVD Compr. (7/128)
abs(original − approximated)
Singular Values (Log−Scale)
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10^4
10^2
10^0
10^{-2}
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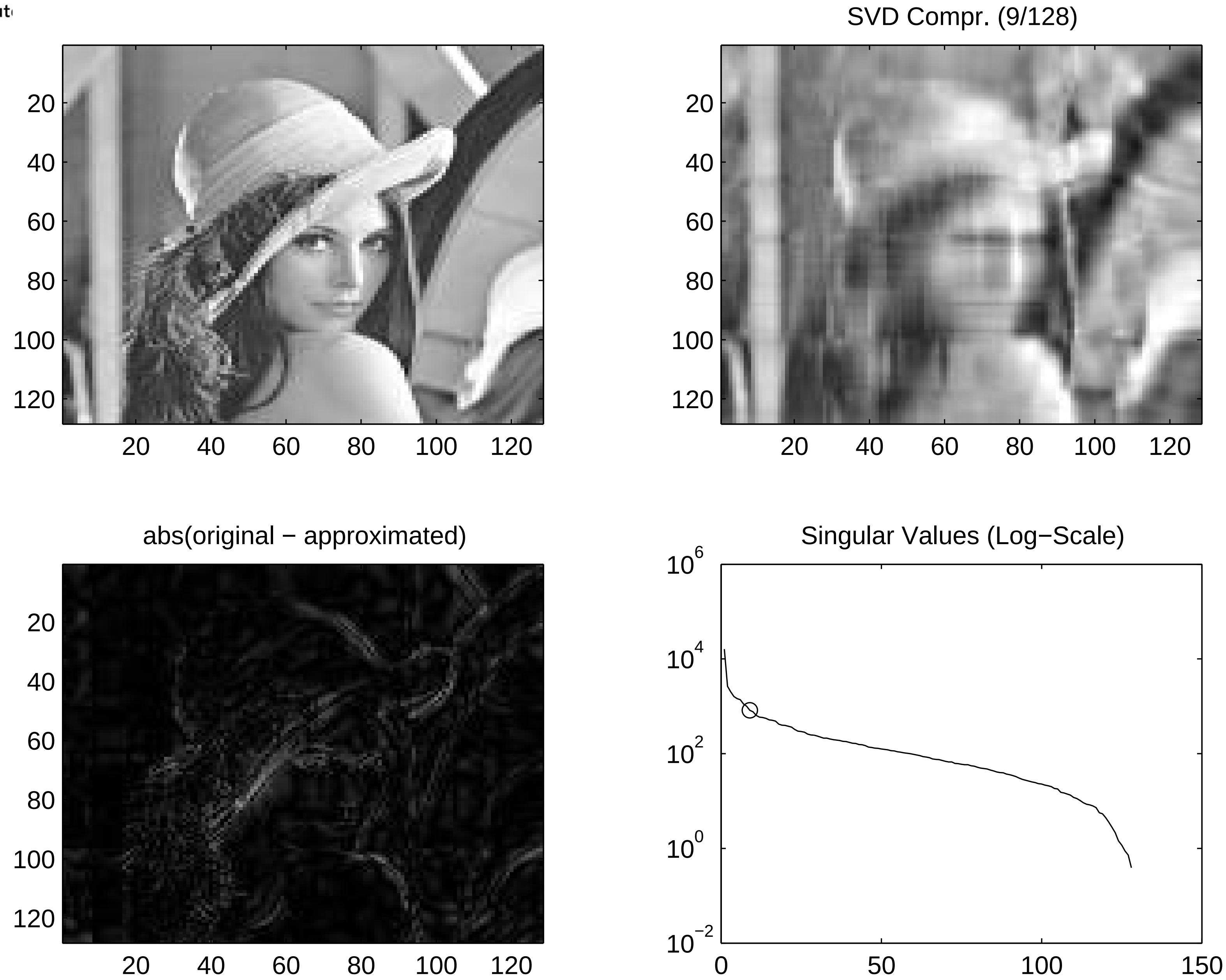
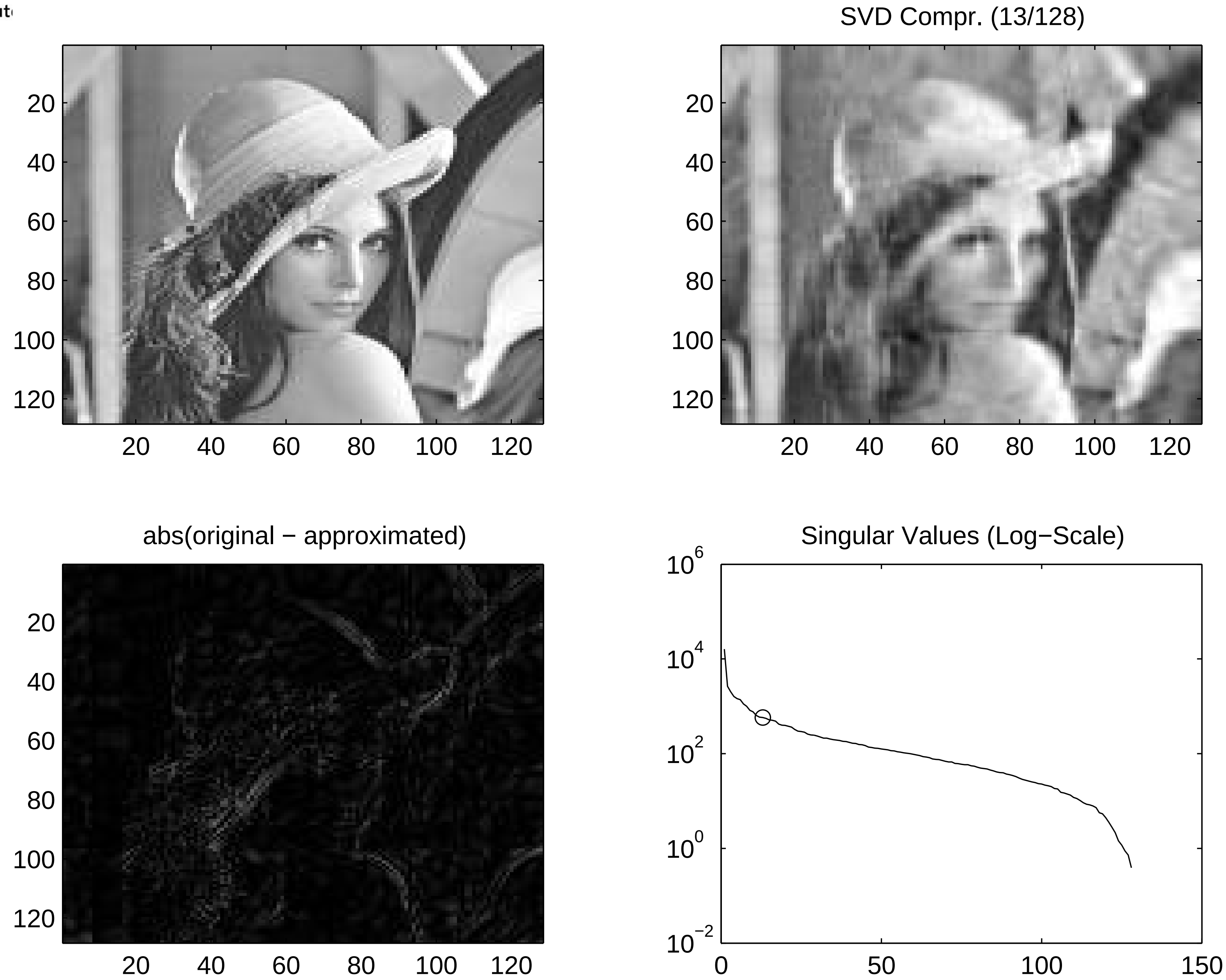
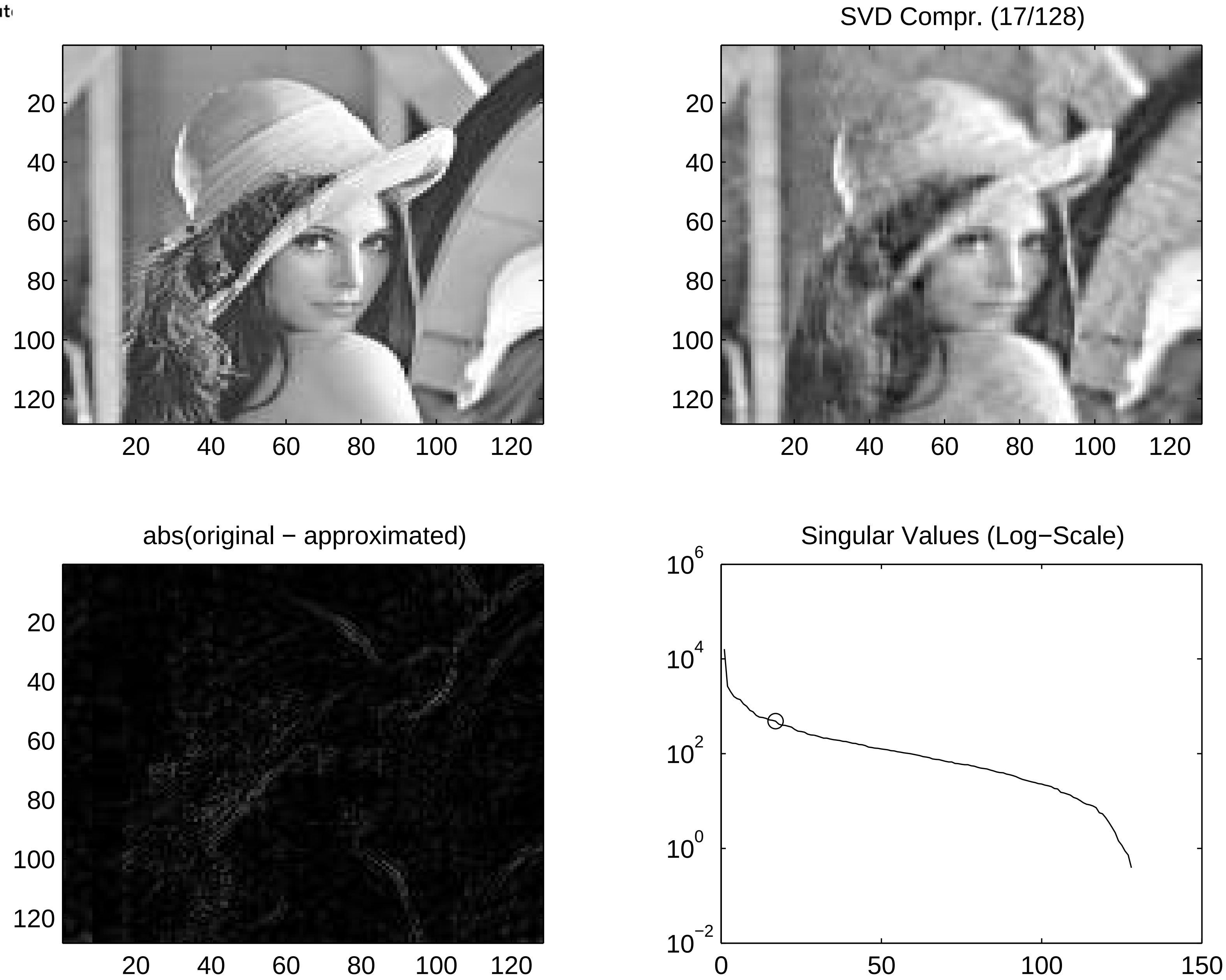

SVD Compr. (9/128)
abs(original − approximated)
Singular Values (Log−Scale)


SVD Compr. (13/128)
abs(original − approximated)
Singular Values (Log−Scale)
20 40 60 80 100 120
10^6 10^4 10^2 10^0 10^−2
0 50 100 150


SVD Compr. (17/128)
abs(original − approximated)
Singular Values (Log−Scale)
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10^{6}
10^{4}
10^{2}
10^{0}
10^{-2}
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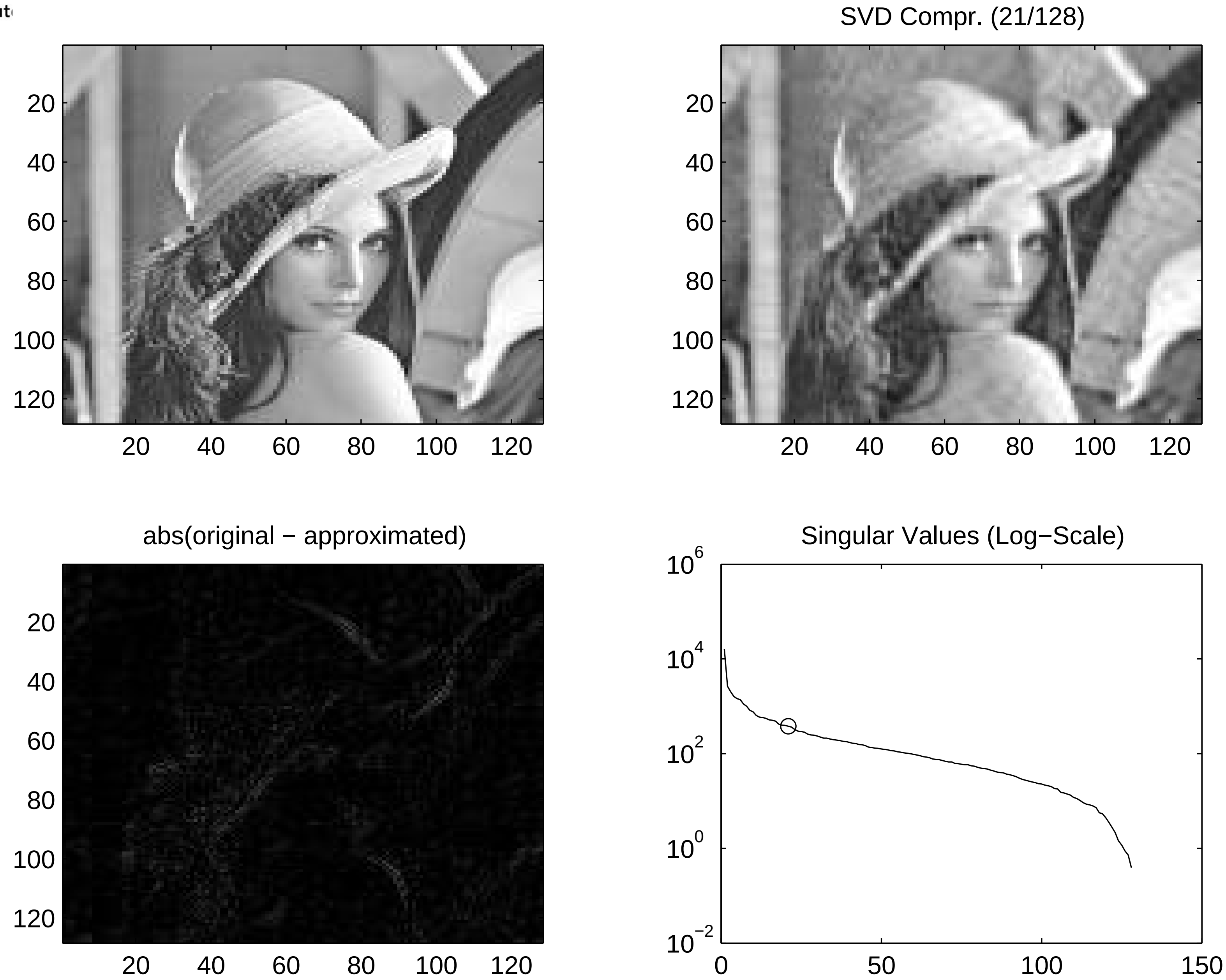
SVD Compr. (21/128)
abs(original − approximated)
Singular Values (Log−Scale)
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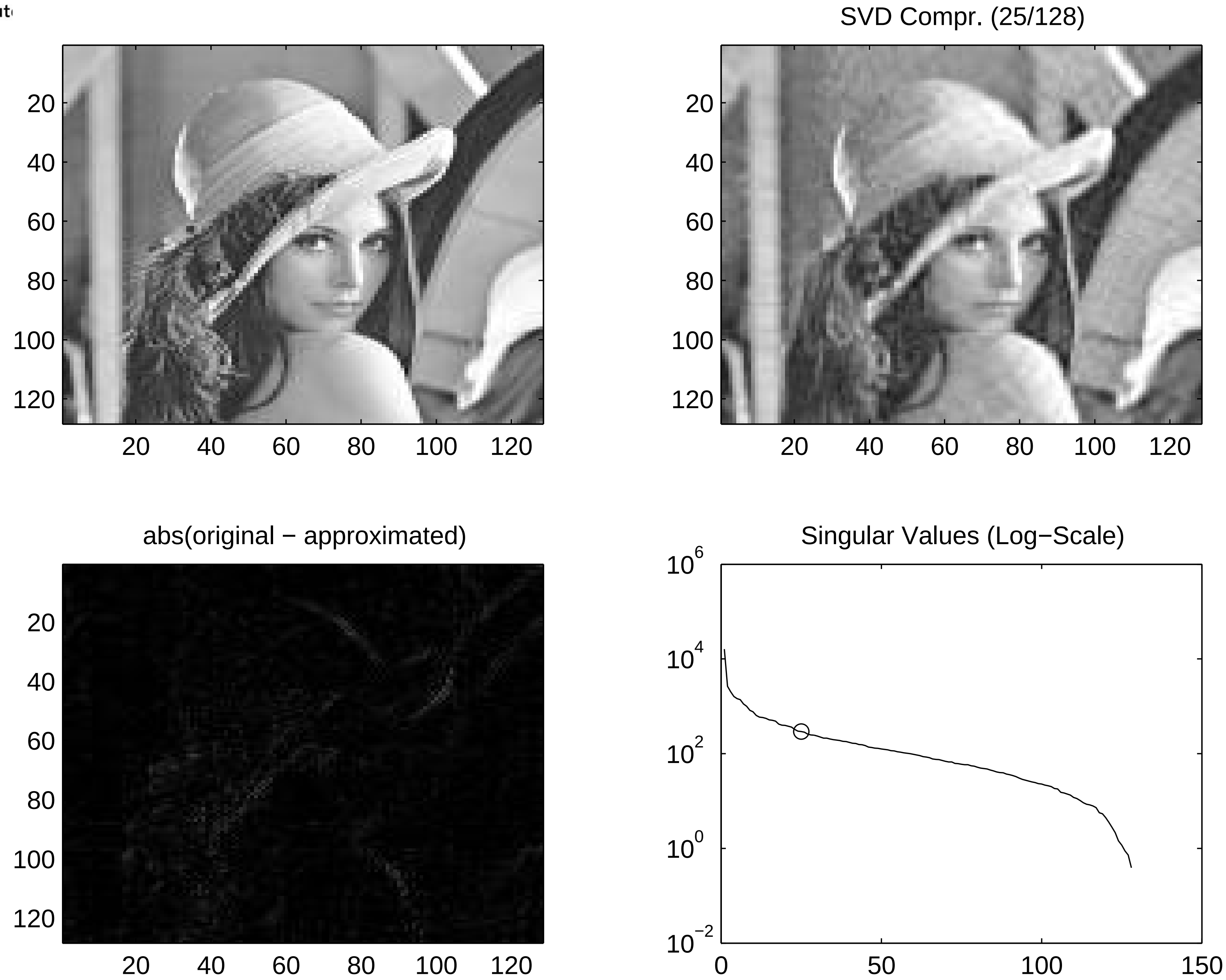
10^{6}
10^{4}
10^{2}
10^{0}
10^{-2}


SVD Compr. (25/128)
abs(original − approximated)
Singular Values (Log−Scale)
20
40
60
80
100
120
10^{6}
10^{4}
10^{2}
10^{0}
10^{-2}
0
50
100
150

SVD Image compression–Color Images

- A color image can be represented using three $m \times n$ matrices A_1, A_2, A_3.
- Project: Compare 2 approaches to SVD compression

Method 1: Do svd compression to all 3 matrices
Method 2: Do svd compression on a single $3m \times n$ block matrix.

$$W = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

Compare quality of images and computational efficiency of methods

Conclusions

- One undergraduate course in linear algebra is not enough.
- Many important topics get left out of the first course.
- A second course is necessary to reinforce material from the first course.
- The first course ends just before one gets to the most interesting and useful topics.
- Linear algebra is an ideal subject for undergraduate research and explorations.