Master Syllabus
Course: MTH 312 Advanced Calculus II
Cluster Requirement: Intermediate Writing 1C

This University Studies Master Syllabus serves as a guide and standard for all instructors teaching an approved course in the University Studies program. Individual instructors have full academic freedom in teaching their courses, but as a condition of course approval, agree to focus on the outcomes listed below, to cover the identified material, to use these or comparable assignments as part of the course work, and to make available the agreed-upon artifacts for assessment of learning outcomes.

Course Overview:

The Advanced Calculus I-II (MTH 311–MTH 312) sequence is where majors study calculus and real analysis in a rigorous manner; that is, these are the courses where majors learn to read and write formal analytic proof. Advanced Calculus II is a continuation of Advanced Calculus I with emphasis on properties of integrals, uniform convergence, and related topics.

During the course students are exposed to historical roots of mathematical concepts which have been refined over many centuries. Student writing will focus on several aspects of technical mathematics:

- The historical context of concepts of analysis and calculus and their definitions – for example, the Cauchy-Riemann integral; uniform and pointwise convergence of sequences of functions.
- Early theorems on, and limitations of, these concepts.
- Recent developments in analytical mathematics.
- Learning to read and write rigorous proofs, with a specific focus on structured proofs.

During this sequence, students are expected to submit well-written homework assignments and projects that are graded by the instructor. Because of the intensive nature of the course and the amount of grading involved, the enrollment will be limited to 20 students. Finally, one of the important goals of this course is to prepare students to be successful in their subsequent 400-level capstone courses.

Grading:

Course grades will be based primarily upon written homework assignments, a written midterm exam and a written final exam or final project. Up to 10% of the grade may be student participation.

Learning Outcomes:

Course-Specific Learning Outcomes:

After successfully completing Advance Calculus II, students will be able to demonstrate improvement in the following four of the Department of Mathematics' six learning outcomes:

1. Content knowledge and skills: Students possess specific technical/analytical skills and conceptual understanding in core areas of mathematics including calculus, linear algebra, combinatorics, differential equations, advanced calculus (analysis) & modern algebra.
3. Mathematical rigor: Students are able to reason rigorously in mathematical arguments. They can follow abstract mathematical arguments and write their own proofs.
4. Communication: Students are able to effectively communicate mathematics: reading, writing, listening, and speaking. Students make effective use of the library, conduct research and make oral and written presentations of their findings.
6. Flexible problem solving: Students are able to transfer the use of facts, concepts, and skills learned in a given context to solve problems in novel settings.

University Studies Learning Outcomes: (Intermediate Writing-IC)

After completing this course, students will be able to:

1. Read with comprehension and critically interpret and evaluate written work in discipline-specific contexts.
2. Demonstrate rhetorically effective, discipline-specific writing for appropriate audiences.
3. Demonstrate, at an advanced level of competence, use of discipline-specific control of language, modes of development and formal conventions.
4. Demonstrate intermediate information literacy skills by selecting, evaluating, integrating and documenting information gathered from multiple sources into discipline-specific writing.

Examples of Advanced Calculus/Real analysis Texts;

Bartle, R. G. (1996). Return to the Riemann integral. *The American Mathematical Monthly*, 103(8), 625-632.

Bartle, R. G. (2001). A modern theory of integration (Vol. 32). Providence: American Mathematical Society.

Bartle, R. G. & Sherbet, D. R. (2011) *Introduction to Real Analysis*, 4th Edition, Wiley,

Brown, D. E. (2013) Introduction to the Cauchy integral (Version 1.2). Dept. of Mathematics, BYU–Idaho.
Burk, F. (2007) *A Garden of Integrals*. Dolciani Mathematical Expositions (Book 31) Mathematical Association of America. Freely available for student use through JSTOR.
Lebl , J. (2014) Basic Analysis: Introduction to Real Analysis, Open Source Text, www.jirka.org/ra
Mawhin, J. (2007). Two histories of integration theory: riemannesque vs romanesque. *Bulletin de la Classe des Sciences de l'Académie Royale de Belgique*, 6(18), 47-63.
Rudin, W. (1964). *Principles of mathematical analysis*. New York: McGraw-Hill.

Examples of Texts Dealing with Mathematical Writing:

Halmos, P. R. (1970). How to write mathematics. *Enseign. Math*, 16(2), 123-152.
Lamport, L. (1995). How to write a proof. *The American mathematical monthly*, 102(7), 600-608.
Leron, U. (1983). Structuring mathematical proofs. *The American Mathematical Monthly*, 90(3), 174-185.
Steenrod, N. E., Halmos, P. R., Schiffer, M. M. & Dieudonne, J. A. (1973) *How to Write Mathematics*. American Mathematical Society, December 1973.
Sterrett, A. (1992). *Using Writing To Teach Mathematics*. MAA Notes, Number 16. Mathematical Association of America, 1529 18th Street NW, Washington, DC 20036.
Higham, N. J. (1998) *Handbook of Writing for the Mathematical Sciences*, 2nd Edition, Society for Industrial and Applied Mathematics; 2 Edition, August 1998.

Example Learning Activities and Assignments:

1. Several reading assignments that expose students to exemplary mathematics exposition from a variety of current and classic sources which form the basis of discussions on the writing of mathematics including alternative proofs of important theorems.

2. Frequent short in-class practice exercises that involve attempting to write a portion or outline of a solution to an analytic problem or a formal proof of a theorem posed by the instructor will also be given. Students review and critique each other's written solutions to the in-class practice exercises.

3. Weekly written homework assignments involving writing complete formal 1–3 page solutions to analytic problems or proofs of theorems. Students use LaTeX to write their assignments.

4. Depending on students' backgrounds and interests, the instructor may have students work in groups on different advanced calculus projects involving proofs and applications of important theorems including projects involving applications of advanced calculus to another discipline.

5. Students maintain an online repository of their written work so that their improvement during the semester can be documented.

Examples of writing instruction:

The list below contains examples of writing instruction related to several core topics for MTH 312. These are examples intended to illustrate the level and depth of writing instruction, and are not intended to be prescriptive as to topic.

Structured proofs will be written as internally hypertext linked LaTeX documents.

<table>
<tr><th>Course topic</th><th>Examples of writing instruction</th></tr>
<tr><td>Recap of structured proofs in an historical context: The Lune of Hippocrates; Eudoxus method of exhaustion; Archimedes method for the area of a parabolic segment.</td><td>Structured proofs (refs: Leron,1983; Lamport,1995) are introduced in MTH 311 Advanced Calculus I. Students read the two referenced articles and then read historical accounts of the topics (opposite) and re-write them as structured proofs.</td></tr>
<tr><td>Introduction to integrals: tagged partitions of intervals</td><td>Write a detailed top level plan to use software to produce a uniformly random, tagged partition of a closed bounded interval with a fixed, but arbitrary, number of partition points, and implement that plan in code.
Write a detailed top level plan to use software to produce a refinement of two given tagged partitions of a closed bounded interval and then implement that plan in code.</td></tr>
<tr><td>Cauchy's formulation of the definite integral.</td><td>Write a detailed coherent account of Cauchy's development of the definite integral, including reference to:
computational methods for computing the Cauchy integral;
a structured proof that the Cauchy integral is unique;
a structured proof of the Cauchy criterion for δ-fine partitions for continuous functions</td></tr>
<tr><td>Riemann's formulation of the definite integral</td><td>Write a detailed coherent account of Riemann's development of the definite integral, including:
reference to the Riemann integral being an improvement of the Cauchy integral;
a structured proof of Riemann's inerrability criterion.
Write structured proofs of the Riemann integrability of monotone bounded, and continuous, functions, defined on a closed bounded interval.</td></tr>
<tr><td>Numerical evaluation of the Riemann definite integral</td><td>Write a detailed top level plan to use software to numerically approximate the Riemann integral of a bounded function on a closed bounded interval to within a specified degree of accuracy, and then implement that plan in code.</td></tr>
<tr><td>Riemann integrals and limits of functions</td><td>Write a structured proof that the integral of the pointwise limit of a sequence of bounded functions on a closed bounded integral, may not be the limit of the integrals of the functions.</td></tr>
</table>

Uniform convergence	• Write a structured proof that the uniform limit of a sequence of continuous functions, defined on a closed bounded interval, is continuous. • Discuss, in detail, an example of a sequence of continuous functions that converges uniformly to a function on a closed bounded interval, for which the Riemann integral does not preserve the uniform limit.
Deficiencies of the Riemann integral	• Discuss in detail, from texts and other sources, deficiencies of the Riemann integral and the need for a more general, well-behaved integral. • Construct a proof of the existence of a function with a bounded derivative whose derivative is not Riemann integrable.
Gauge integral (aka Henstock-Kurzweil integral)	Write an account, with selected proofs, of how the gauge interval, as defined by Henstock and Kurzweil, remedies many deficiencies of the Riemann integral, with reference to: • a structured proof that the Dirichlet function is gauge integrable but not Riemann integrable; • integration of unbounded functions; • structured proof of Cousin's lemma on the existence of δ-fine partitions for a gauge δ , using the Nested Intervals theorem; • structured proof of the gauge integrability of step functions; • definition of "almost everywhere" and proof that a function that is 0 almost everywhere is gauge inferable with integral 0.; • the Cauchy-criterion for gauge integrability; • monotone convergence theorem for the gauge integral.
Measurable sets and measure	Write proofs of the basic properties of the measure of measurable sets from the gauge integral.
The Cantor set	Write a detailed coherent account of the definition of the middle-thirds Cantor set including: • characterization of elements in the Cantor set by their base 3 representation; • a structured proof that the Cantor set is uncountable; • a structured proof that the Cantor set has measure zero.

University Studies Intermediate Writing course criteria

1. *Intermediate Writing courses employ writing as a method for deepening student learning. Assignments in the course ask students to practice working in writing with concepts central to the course content. Intermediate Writing courses also often use informal or "low-stakes" writing exercises, pre-writing exercises, and other "best practices" (such as peer review and online written discussion) to help students recursively practice concepts and problem-solving methods before asking them to turn in formal written work for a final grade*.

Writing exercises in MTH 312 are primarily based around two distinct forms of writing, both critical for mathematical development:

- Writing proofs.
- Writing clear syntheses of conceptual developments.

The writing of proofs in MTH 312 Advanced Calculus II is based on a form of writing called "structured proofs" dealt with, in detail in the reference material:

> Lamport, L. (1995). How to write a proof. *The American Mathematical Monthly*, 102(7), 600-608.
> Leron, U. (1983). Structuring mathematical proofs. *The American Mathematical Monthly*, 90(3), 174-185.

An essential aspect of a structured proof is that it contains a top level overview of the central idea or ideas of the proof, yet leaves lower level details to be filled in, successively, as appendices to the main top level argument. Students have been introduced to this style of proof writing in MTH 311 Advanced Calculus I, where they write structured proofs as internally hypertext linked documents using LaTeX. The nature of a structured proof requires knowing when to stop the level of detailed explanation, which involves having a clear understanding of one's audience. As part of learning to write proofs, students need to come to an understanding of their intended audience, and the levels of explanation likely to be required by that audience. This is a critical aspect of the development of mathematical maturity that is not addressed simply by learning content.

Writing correct and comprehensible proofs in mathematics is a learned skill – no one is born knowing how to do it – that is critical to future mathematical success and that manifests as a developmental process, beginning in MTH 180, MTH 181, Discrete Structures I & II, continuing into MTH 221 Linear Algebra, through to MTH 311 & MTH 312 Advanced Calculus I & II. Proof writing necessarily begins with an idea and it is refining and turning that idea into a reasoned mathematical argument, capable of withstanding the highest level of critical examination, that constitutes proof writing. Proofs are expected at various levels of difficulty and, as is common in all mathematical writing, more complicated proofs are usually broken down into a digestible sequence of preliminary results, often referred to as "lemmas" or "propositions". Students will receive weekly practice in proof writing and in-class practice at providing and receiving friendly critical comments.

Another major aspect of mathematical writing is writing syntheses, or reviews, of central mathematical ideas. Several examples of this are presented in the table above. Writing syntheses is important in helping students pull their thoughts together in a clear and coherent exposition of what is, in general, a conceptually sophisticated topic.

2. *Faculty provide feedback, on-going guidance, and clear expectations for "effective" written response. Intermediate Writing courses enable students to write as "apprentices" in the field; faculty act as writing mentors and disciplinary "experts." The exact nature of this relationship will vary from class to class. Best practices include: holding one-on-one or group conferences about student written work, commenting on early drafts before papers are submitted for a final grade, peer review and response activities, discussions and modeling of techniques for drafting (organizing content, integrating sources, contextualizing the ongoing conversation of the field, assessing the type and amount of information readers need, making appropriate grammatical and stylistic choices), breaking down writing assignments into manageable chunks for students, and giving feedback to students about their writing throughout the writing process*.

No one is born knowing how to write proofs, and the writing of proofs is an activity based on the existence of relevant mathematical ideas. As Alexandre Grothendick so eloquently wrote in 1983:

> "What my experience of mathematical work has taught me again and again, is that the proof always springs from the insight, and not the other way round – and that the insight itself has its source, first and foremost, in a delicate and obstinate feeling of the relevant entities and concepts and their mutual relations. The guiding thread is the inner coherence of the image which gradually emerges from the mist, as well as its consonance with what is known or foreshadowed from other sources – and it guides all the more surely as the "exigence" of coherence is stronger and more delicate."

For students, writing proofs is a developmental process that is supported from their freshman year on. In MTH 312 Advanced Calculus II, students are learning to be more precise and rigorous with their proofs, yet to do this well they need considerable in-class support, both from the instructor and from each other, in preparing an initial proof draft and then having that draft iteratively critiqued. Our experience is that students need considerable support in getting initial ideas into a coherent proof format, and need success in building confidence to find their own authentic voice and not simply parrot what they imagine makes the instructor happy.

3. *As detailed above, writing about Advanced Calculus accounts for at least 90% of the course grade. Thus, a significant portion of the graded work for the course should be writing about the course material. Students must complete all writing assignments to pass the course.*

4. *Students must complete at least 20 pages of writing. Informal (pre-writing, low stakes, and freewriting) and formal writing may be included in this final count. The type of writing assigned will be determined by the course content, and may include analytic/persuasive/critical*

essays and reports, white pages, reviews, journals, proposals, lab reports/observations, written responses to readings, application of key concepts, reflections on their own written work. This page count does not include in-class exams or drafts of final papers.

Students will be given writing assignments each week of the semester. These writing assignments will entail approximately 2-3 pages each week. Over a 14-week semester this entails a total of about 30-40 pages of written work.